Professor Dr. Jens Vygen
Matthias Kaul

Exercise Set 11

Exercise 11.1. Let $f : 2^U \to \mathbb{R}$ be a submodular function and R be a random subset of U where every element is chosen independently with probability $\frac{1}{2}$. Show that

$$\mathbb{E}[f(R)] \geq \frac{1}{4}(f(\emptyset) + f(A) + f(U \setminus A) + f(U))\ \forall A \subseteq U,$$

and conclude that for non-negative f we get

$$\mathbb{E}[f(R)] \geq \frac{1}{4} \max_{A \subseteq U} f(A).$$

That is to say, sampling a random subset of U in the above way is a randomized 4-approximation algorithm for Submodular Function Maximization.

(4 points)

Exercise 11.2. Recall the randomized algorithm for Submodular Function Maximization presented in the lecture. We may derandomize it by changing the update step for A, B to setting $A := A \cup \{i\}$ if $\Delta_A \geq \Delta_B$, and $B := B \setminus \{i\}$ otherwise. Show that this algorithm yields a 3-approximation deterministically in linear time.

Note: This is a weaker derandomization than the one presented in the lecture.

(5 points)

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Exercise 11.3. Consider the following variant of the UNCAPACITATED FACILITY LOCATION PROBLEM: We are given sets of clients D and facilities F, along with profits $p : D \times F \to \mathbb{R}_{\geq 0}$ and uniform facility opening costs c with

$$0 \leq c \leq \frac{1}{|F|} \sum_{d \in D} \max_{x \in F} \{p(d, x)\}.$$

The goal is to compute a set X of facilities to open up such that the net profit is maximized, that is we want to maximize the following function:

$$f(X) := \sum_{d \in D} \max\{0, \max_{x \in X} p(d, x)\} - c \cdot |X| \text{ where } X \subseteq F.$$

- Prove that the objective function f is submodular.
- f is not nonnegative in general, but the algorithms in the lecture and om Exercise 11.2 still obtain the claimed approximation guarantees. Why?

(4 points)

Deadline: Jan 7$^{\text{th}}$, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws24/co_exercises_ws.html`

In case of any questions feel free to contact me at mkaul@uni-bonn.de.