Exercise Set 9

Exercise 9.1. Given an instance of Path TSP as a complete graph G with a metric cost function $c : E(G) \to \mathbb{R}_{\geq 0}$ and two fixed vertices $s, t \in V(G)$, we define the following LP:

$$\begin{aligned} &\min c(x) && && \\ &\text{subject to} && x(\delta(U)) \geq 2 && (\emptyset \neq U \subset V(G), |U \cap \{s,t\}| \text{ even}) \\ & && x(\delta(U)) \geq 1 && (\emptyset \neq U \subset V(G), |U \cap \{s,t\}| \text{ odd}) \\ & && x(\delta(v)) = 2 && (v \in V(G) \setminus \{s,t\}) \\ & && x(\delta(v)) = 1 && (v \in \{s,t\}) \\ & && x_e \geq 0 && (e \in E(G)) \end{aligned} \tag{PTSP-LP}$$

Every integral LP solution x is the incidence vector of a Hamiltonian s-t-path. Show that PTSP-LP has integrality gap at least $\frac{3}{2}$.

(4 points)

Exercise 9.2. Let x be any feasible solution to PTSP-LP. Prove that x can be written as a convex combination of incidence vectors of spanning trees of G.

(4 points)

Exercise 9.3. Given an instance of (G, c, s, t) of Path TSP show that Christofides' Algorithm (for Path TSP) computes a solution of cost at most $5/3$ times the value of PTSP-LP.

Hint: Recall Wolsey's analysis of the integrality gap of the Subtour-Elimination LP. To bound the cost of a spanning tree (V, S), use Exercise 9.2. To bound the cost of the $odd(S)\Delta\{s,t\}$-join show that the vector $\frac{1}{3}(x^* + \chi_S)$ is in the $odd(S)\Delta\{s,t\}$-join polytope where x^* is a PTSP-LP solution and χ_S is the incidence vector of S.

(5 points)

Exercise 9.4. Fix a graph G and recall the formulation for the Spanning-Tree Polytope given in the lecture,

$$P = \{x \in \mathbb{R}^E \mid x_e \in [0,1], \sum_{e \in E} x_e = |V| - 1, \sum_{e \in E(G[X])} x_e \leq |X| - 1 \ \forall \emptyset \neq X \subset V\}.$$

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Give a polynomial-time separation oracle for P, i.e. an algorithm that given some $x \in \mathbb{R}^E$ in polynomial time either decides $x \in P$ or returns a constraint of P that is violated by x. You may not use the equivalence of separation and optimization.

Hint: You need to compute a set X minimizing $\sum_{e \notin E(G[X])} x_e + |X|$. Encode this problem as a directed minimum s-t-cut problem in some modified graph.

(6 points)

Deadline: Dec 10th, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws24/co_exercises_ws.html`

In case of any questions feel free to contact me at mkaul@uni-bonn.de.