Linear and Integer Optimization
Assignment Sheet 8
Inofficial English Translation

1. Let G be simple undirected graph. Consider the following linear program:

$$\begin{array}{llll} \min & \sum_{e=\{v,w\}\in E(G)} x_{vw} & & \\ \text{s.t.} & \sum_{w\in S} x_{vw} \geq \lceil \frac{1}{4}|S|^2 + \frac{1}{2}|S| \rceil & & \text{for } v \in V(G), S \subseteq V(G)\setminus\{v\} \\ & x_{uw} \leq x_{uv} + x_{vw} & & \text{for } u,v,w \in V(G) \\ & x_{vw} \geq 0 & & \text{for } v \in V(G) \\ & x_{vv} = 0 & & \text{for } v \in V(G) \end{array}$$

   (a) Show that this is a relaxation of the following problem: Find distances x_{vw} for the nodes of G such that $\sum_{e=\{v,w\}\in E(G)} x_{vw}$ is minimized under the condition that there is an ordering $\{v_1,\dots,v_{|V(G)|}\} = V(G)$ with $x_{v_iv_j} = |i-j|$ for $i,j \in \{1,\dots,|V(G)|\}$.

   (b) Prove that there is a polynomial-time separation oracle for the polyhedron of the feasible solutions of the LP. (2+3 points)

2. A semidefinite program is an optimization problem

$$\begin{aligned} \min\ & C \star X & & \\ & A_i \star X \leq b_i & & \forall i = 1,\dots,m \\ & X \succeq 0 & & \\ & X \in \mathbb{R}^{n\times n} & & \end{aligned}$$

   where $C, A_1, \dots, A_n$ are matrices, $A \star X := \sum_{1\leq i,j\leq n} a_{ij}x_{ij}$ and $X \succeq 0$ means that X is symmetric and positiv semidefinite.

   (a) Show that the set $\{X \in \mathbb{R}^{n\times n} \mid X \succeq 0\}$ is a closed cone.

   (b) Construct a polynomial-time separation oracle for this set. (You may assume that you can compute basic arithmetic operations on real numbers, including square roots, exactly and in constant time.) (3+3 points)

3. Let $\mathcal{A}$ be an algorithm that finds, given a feasiable and bounded LP $\max\{c^tx \in \mathbb{R}^n \mid Ax \leq b\}$ (with $c \in \mathbb{Q}^n$, $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{Q}^m$) in polynomial running time an optimum solution. Show that there is a polynomial algorithm that always finds an optimum solution that is a vertex of $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, provided that P is pointed. (4 points)

4. Let $K \subseteq \mathbb{R}^n$ be a convex set with $rB^n \subseteq K \subseteq RB^n$ for some numbers $0 < r \leq \frac{R}{2}$. Assume that you are given an oracle with polynomial running time that computes an optimum solution in K for any linear objective function. Show that there is a separation oracle with polynomial running time for $K^* := \{y \in \mathbb{R}^n \mid y^tx \leq 1 \text{ for all } x \in K\}$. (5 points)

Due date: Thursday, June 2, 2022, before the lecture in the lecture hall.