Research Institute for Discrete Mathematics
Universität Bonn

Linear and Integer Optimization
Assignment Sheet 6
Inofficial English Translation

1. Consider the following linear program:

$$\begin{array}{lrcrcrcrcrcrcl}
\max & 2.3x_1 & + & 2.15x_2 & - & 13.55x_3 & - & 0.4x_4 & & & & & & \\
\text{s.d.} & 0.4x_1 & + & 0.2x_2 & - & 1.4x_3 & - & 0.2x_4 & + & x_5 & & & = & 0 \\
 & -7.8x_1 & - & 1.4x_2 & + & 7.8x_3 & + & 0.4x_4 & & & + & x_6 & = & 0 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0
\end{array}$$

   Use $\{5, 6\}$ as an initial basis for the Simplex Algorithm. As pivot rules, choose the new basis index α such that r_α maximized. Let the index β that leaves the basis be chosen such that it minimizes (among all candidates) the value $q_{\beta\alpha}$. Show that the Simplex Algorithm does not terminate under these considitions. (5 points)

2. Consider a linear program in appropriate form for the Simplex Algorithm, i.e. $\max\{c^t x \mid Ax = b, x \geq 0\}$ such that $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) = m$ and $Ax = b$ is feasible. Prove or disprove the following statements:

   (a) A variable that has just entered the basis can leave the basis in the next iteration.

   (b) A variable that has just left the basis can enter the basis in the next iteration.

   (c) If x is unique optimum basic solution and $\tilde{x}$ a second best basic solution with strictly smaller solution value then x can be computed from $\tilde{x}$ by exchanging one basic variable.

   (d) If no basic solution is degenerated and the LP is feasible and bounded then there is a unique optimum solution. (1+1+1+1 points)

3. Consider a linear program $\max\{c^t x \mid Ax = b, x \geq 0\}$. Let B be a feasible basis with basic solution x^* and reduced cost vector $r \leq 0$ (so x^* is an optimum solution). Let $I = \{j \in N \mid r_j = 0\}$.

   (a) Prove that x^* is the unique optimum solution if $I = \emptyset$.

   (b) Assume that $I \neq \emptyset$. Prove that in this case x^* is the unique optimum solution if and only if the following linear program has the optimum solution value 0:

$$\begin{array}{lrcll}
\max & \sum_{i \in I} x_i & & & \\
\text{s.t.} & Ax & = & b & \\
 & x_i & = & 0 & \text{for } i \in N \setminus I \\
 & x_i & \geq & 0 & \text{for } i \in B \cup I
\end{array}$$

(2+3 points)

p.t.o.

4. Consider a linear program $\max\{c^t x \mid Ax = b, x \geq 0\}$ such that $A \in \mathbb{R}^{m \times n}$, $\operatorname{rank}(A) = m$ and $Ax = b$ is feasible. Let B be a dual feasible basis, i.e. a basis such that $\tilde{y} = (A_B^t)^{-1} c_B$ is a feasible solution of the dual LP.

   (a) Show that the entry z_0 of the simplex tableau $T(B)$ is the cost of the dual solution.

   (b) Let $\beta \in B$ with $p_\beta < 0$ and $\alpha \in N$ with $q_{\beta\alpha} > 0$ such that $\frac{r_\alpha}{q_{\beta\alpha}} \geq \frac{r_j}{q_{\beta j}}$ for all $j \in N$ with $q_{\beta j} > 0$. Prove that $(B \setminus \{\beta\}) \cup \{\alpha\}$ is a dual feasible basis. Moreover, show that the value of the dual solution is changed by $\frac{-p_\beta}{q_{\beta\alpha}} r_\alpha$. (2+4 points)

Due date: Thursday, May 19, 2022, before the lecture in the lecture hall.