Combinatorics, Graphs, Matroids

July 4, 2016
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`http://www.or.uni-bonn.de/lectures/ss16/kgm_ss16.html`

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1 Introduction

1.1 Fundamental Counting Rules

(I) For pairwise disjoint finite sets $U_1, \dots, U_k$, we have $|\bigcup_{i=1}^n U_i| = \sum_{i=1}^n |U_i|$.

(II) For finite sets $U_1, \dots, U_k$, we have $|U_1 \times \cdots \times U_n| = \prod_{i=1}^n |U_i|$.

(III) If there is a bijection between two sets S and T, then $|S| = |T|$.

Examples:

We call a set an n-set if it is a finite sets with $n \in \mathbb{N}$ elements (where $\mathbb{N}$ includes 0).

Proposition 1 *For two n-sets A and B, the number of bijections from A to B is $n!$.*

Proof: Let U be the set of bijections from A to B. Apply induction in n. The case $n = 0$ is trivial because then $0! = 1 = |U|$. Let $n > 0$ and $x \in A$. For $i \in B$, let U_i be the set of all bijections f from A to B with $f(x) = i$. Then by induction $|U_i| = (n-1)!$ for $i \in B$. Moreover, we have $U = \dot\bigcup_{i \in B} U_i$, so by (I) we get $|U| = \sum_{i \in B} |U_i| = n \cdot (n-1)! = n!$. $\square$

Corollary 2 *The number of permutations of an n-set is $n!$.* $\square$

Proposition 3 *The number of mappings from a k-set A to an n-set B is n^k.*

Proof: There is a bijection between the set of all mappings from A to B and the set $\underbrace{B \times \cdots \times B}_{k \text{ times}}$. Thus by (III) and (II), the number of such mappings is $|B|^k = n^k$. $\square$

Notation: We denote the set of all mappings from A to B by B^A.

Proposition 4 *The number of subsets of an n-set is 2^n.*

Proof: For an n-set A, there is a bijection between its power set (i.e. the set of its subsets) and the set of mappings from A to $\{0,1\}$: For each $B \subseteq A$ define a mapping $f_B : A \to \{0,1\}$ by setting $f_B(x) = 1$ if and only if $x \in B$. Then, $B \mapsto f_B$ is clearly a bijection. Thus, the statement follows from (III) and the previous proposition. $\square$

Notation: We denote the power set of a set A by 2^A.

What does counting mean? Possible answers can be:

(i) direct, closed formula

(ii) sum

(iii) recursive formula

Example: Consider the number y_n of 0-1-2-strings of length n with even number of 1s and odd number of 2s. Then, one easily gets an answer of type (ii): $y_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} 2^{2i}$. Also a recursive solution can be found easily: $y_n = 3^{n-1} - y_{n-1}$ for $n \in \mathbb{N} \setminus \{0\}$ (see exercises for the correctness of the formulas). We will examine methods to transform such sum and recursive formulas into a closed formula.

1.2 Elementary Counting Coefficients

The most imporant counting coefficient is the **binomial coefficient** $\binom{n}{k}$ which is defined as the number of k-subsets of an n-set (for $k, n \in \mathbb{N}$).

Proposition 5 *Let $n, k \in \mathbb{N}$ with $\leq k \leq n$. Then:*

(a) $\binom{n}{k} = \binom{n}{n-k}$.

(b) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ *for* $k \geq 1$.

(c) $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Proof:

(a) Trivial.

(b) Let M by an n-set and let $x \in M$ an arbitrary element. Then

$$\begin{aligned}
\binom{n}{k} &= |\{N \in 2^M \mid |N| = k\}| \\
&= |\{N \in 2^M \mid |N| = k, x \in N\} \dot{\cup} \{N \in 2^M \mid |N| = k, x \notin N\}| \\
&= |\{N \in 2^M \mid |N| = k, x \in N\}| + |\{N \in 2^M \mid |N| = k, x \notin N\}| \\
&= \binom{n-1}{k-1} + \binom{n-1}{k}.
\end{aligned}$$

(c) Induction in $n+k$. $n+k=0$ is trivial, so assume $n+k>0$.

$$\begin{aligned}
\binom{n}{k} &= \binom{n-1}{k-1} + \binom{n-1}{k} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\
&= \frac{(n-1)!(k+n-k)}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!}.
\end{aligned}$$

□

Pascal's triangle (see p. 19 from Aigner [2007])

Sums in Pascal's triangle:

- Column sums: $\sum_{m=0}^{n} \binom{m}{k} = \binom{n+1}{k+1}$ for $n, k \in \mathbb{N}$.
  For $m \in \{0, \dots, n\}$, $\binom{m}{k}$ is the number of ways to choose $k+1$ numbers from the set $\{1, \dots, n+1\}$ under the condition that $m+1$ is the largest chosen number.
- Diagonal sums: $\sum_{k=0}^{n} \binom{m+k}{k} = \binom{m+n+1}{n}$ for $m, n \in \mathbb{N}$.
  For $k \in \{0, \dots, n\}$, $\binom{m+k}{k}$ is the number of ways to choose n numbers from the set $\{1, \dots, m+n+1\}$ under the condition that $m+k+1$ is the largest number that is not chosen.

Proposition 6 *(Binomial theorem): For $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$, we have:*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof: The coefficient of $x^k y^{n-k}$ in the sum is the number of ways to choose the term x from k of the n factors $(x+y) \cdot \dots \cdot (x+y)$. □

Definition 1 *For $n, k \in \mathbb{N}$ let $S_{n,k}$ be the number of ways to partition an n-set into k non-empty sets. The numbers $S_{n,k}$ are called* **Stirling numbers of the second kind**.

In particular for $n > 0$: $S_{n,0} = 0$, $S_{n,1} = 1$, $S_{n,2} = 2^{n-1} - 1$.

Proposition 7 *For $n, k \in \mathbb{N} \setminus \{0\}$ we have*

$$S_{n,k} = S_{n-1,k-1} + kS_{n-1,k}.$$

Proof: Let M be an n-set and $x \in M$. The set of partitions of M in k subsets can be decomposed in the set of partitions with the set $\{x\}$ (there are $S_{n-1,k-1}$ of them) and the set of partitions where x is an element in a set with more than one element (there are $kS_{n-1,k}$ of them). $\square$

Stirling's triangle of the second kind: see Aigner [2007], p. 21.

Definition 2 *For $n, k \in \mathbb{N}$, let $P_{n,k}$ be the number of ways to write n as the sum of k positive integers (without considering the order of the summands).*

For example $P_{6,3} = 3$ because $6 = 4 + 1 = 3 + 2 + 1 = 2 + 2 + 2$ can be written in three different ways as a sum of three positive integers.

For $n > 0$, we have $P_{n,0} = 0$, $P_{n,1} = 1$, $P_{n,2} = \lfloor \frac{n}{2} \rfloor$.

We can also consider *ordered* partitions of numbers. For example there are 6 ordered 3-partitions of 5: $3 + 1 + 1$, $1 + 3 + 1$, $1 + 1 + 3$, $2 + 2 + 1$, $2 + 1 + 2$, and $1 + 2 + 2$.

Theorem 8 *The number of ordered partitions of a positive integer n into k summands is $\binom{n-1}{k-1}$.*

Proof: Any positive integer number n can be written as a sum of n ones, and these ones can be partitioned into subsequences with lengths $x_1, \ldots, x_k$:

$$n = \overbrace{\underbrace{1 + \cdots + 1}_{x_1} \oplus \underbrace{1 + \cdots + 1}_{x_2} \oplus \cdots \oplus \underbrace{1 + \cdots + 1}_{x_k}}^{n}$$

There is a one-to-one correspondence between the partitionings of the ones into subsequences and the ordered k-partitions of n. Thus, each ordered k-partition of n is given by the choice of the $\oplus$-signs. There are $\binom{n-1}{k-1}$ ways to choose the $k - 1$ $\oplus$-signs, which implies that also the number of ordered k-partitions of n is $\binom{n-1}{k-1}$. $\square$

1.3 Assignments

Observations:

- The number of *surjective* mappings from an n-set to an r-set is $r!S_{n,r}$. To see this, note that there are $S_{n,r}$ ways to partition an n-set into r preimages, and for each such partition we get $r!$ ways to assign the preimages to the elements of an r-set.

- The number of *injective* mappings from an n-set to an r-set is

$$r^{\underline{n}} := r(r-1)\dots(r-n+1).$$

We call $r^{\underline{n}}$ the n-th **falling factorial** of r.

Analogously, we define the n-th **rising factorial** of r as

$$r^{\overline{n}} := r(r+1)\dots(r+n-1).$$

Definition 3 *A k-**multiset** over an n-set A is a set $B \subseteq A \times \mathbb{N}$ such that $\sum_{(a,i)\in B} i = k$.*

We can consider multisets as sets that can contain the same element more than once. Thus we write k-multiset as $\{a_1, \dots, a_k\}$ but in this case there can be numbers $i, j \in \{1, \dots, k\}$ with $i \neq j$ but $a_i = a_j$.

Proposition 9 *The number of k-multisets over an n-set is $\frac{n^{\overline{k}}}{k!} = \binom{n+k-1}{k}$.*

Proof: There is a bijection f between the set of all k-multisets over $\{1, \dots, n\}$ and the set of all k-subsets of $\{1, \dots, n+k-1\}$: For a k-multiset $\{a_1, \dots, a_k\}$ with $a_1 \le a_2 \le \dots \le a_k$ define $f(\{a_1, \dots, a_k\})$ as $\{a_1, a_2+1, a_3+2, \dots, a_k+k-1\}$. It is easy to check that this is indeed a bijection between the two sets. Since there are $\frac{n^{\overline{k}}}{k!} = \binom{n+k-1}{k}$ k-subsets of $\{1, \dots, n+k-1\}$, this proves the proposition. $\square$

Proposition 10 *For $n, r \in \mathbb{N}$, we have*

$$r^n = \sum_{k=0}^{n} S_{n,k} r^{\underline{k}}.$$

Proof: For two sets A and B, let $\mathrm{Surj}(A, B)$ be the set of surjective mappings from A to B. Let N be an n-set and R and r-set. Then the number of mappings from N to R is r^n, so

$$r^n = \sum_{A\subseteq R} |\mathrm{Surj}(N, A)| = \sum_{k=0}^{r} \sum_{A\subseteq R, |A|=k} |\mathrm{Surj}(N, A)| = \sum_{k=0}^{r} \binom{r}{k} k! S_{n,k} = \sum_{k=0}^{r} S_{n,k} r^{\underline{k}} = \sum_{k=0}^{n} S_{n,k} r^{\underline{k}}$$

For the last equation, we made use of the fact that $S_{n,k} = 0$ for $k > n$ and $r^{\underline{k}} = 0$ for $k > r$. $\square$

Assume that we want to count the ways to assign a set of n ball to r bins. We may or may not be able to distinguish the the balls and we may or may not be able to distinguish the bins. Table 1 gives an overview of the number of assignments.

elements distinguishable		Mapping $f : N \to R$			
N	R	arbitrary	injective	surjective	bijective
Yes	Yes	r^n	$r^{\underline{n}}$	$r! S_{n,r}$	$0 \quad (n \neq r)$ $n! \quad (n = r)$
No	Yes	$\frac{r^{\overline{n}}}{n!}$	$\binom{r}{n}$	$\binom{n-1}{r-1}$	$0 \quad (n \neq r)$ $1 \quad (n = r)$
Yes	No	$\sum_{k=1}^{r} S_{n,k}$	$0 \quad (r < n)$ $1 \quad (r \geq n)$	$S_{n,r}$	$0 \quad (n \neq r)$ $1 \quad (n = r)$
No	No	$\sum_{k=1}^{r} P_{n,k}$	$0 \quad (r < n)$ $1 \quad (r \geq n)$	$P_{n,r}$	$0 \quad (n \neq r)$ $1 \quad (n = r)$

Tabelle 1: Number of mappings from an n-set N to an r-set R with and without additional constraints.

Remarks on Table 1: Most of the entries are easy consequences of previous results. Assume that we can distinguish the elements of R but not the elements of N. Then, the number of all mapping from N to R is $\frac{r^{\overline{n}}}{n!}$ because this is the number of n-multisets over an r-set. The number of surjective mappings from N to R is the number of ordered partitions of the number n into r summands, hence it is $\binom{n-1}{r-1}$.

1.4 Generalized Counting Coefficients

We generalize the definitions of the rising and falling factorials by setting for $r \in \mathbb{C}$ and $k \in \mathbb{N}$:

$$r^{\underline{k}} := r(r-1)(r-2)\dots(r-k+1)$$

and

$$r^{\overline{k}} := r(r+1)(r+2)\dots(r+k-1)$$

We also generalize the binomial coefficients, by setting for $r \in \mathbb{C}$ and $k \in \mathbb{Z}$:

$$\binom{r}{k} := \begin{cases} \frac{r^{\underline{k}}}{k!} & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 \end{cases}$$

The recursive formula from Proposition5(b) can now be generalized to $r \in \mathbb{C}$ and $k \in \mathbb{Z}$:

$$\binom{r}{k} = \binom{r-1}{k-1} + \binom{r-1}{k} \tag{1}$$

For a proof, we can simply use the definition of $\binom{r}{k}$ but we can also apply the "polynomial method": For fixed $k > 0$ (the case $k \leq 0$ is easy), both sides of equation (1) are polynomials in r of degree k. Since we have proved (1) for any positive integer r, we know that both polynomials have the same values at an infinite number of values of r. This means they must be identical. This proves (1).

Proposition 11 *(Vandermonde identity): For $x, y \in \mathbb{C}$ and $n \in \mathbb{N}$, we have:*

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k}$$

Proof: Again, we can use the polynomial method: For $x, y \in \mathbb{N}$, this is true because both terms describe the number of k-subsets of an $s+t$-set. For general values of x and y, the statement follows because both terms are polynomials of degree n and are identical on an infinite number of points. □

Proposition 12 *For $r \in \mathbb{C}$ and $k \in \mathbb{Z}$:*

$$\binom{-r}{k} = (-1)^k \binom{r+k-1}{k}.$$

Proof: We have $(-r)^{\underline{k}} = (-r)(-r-1)\dots(-r-k+1) = (-1)^k r(r+1)\dots(r+k-1) = (-1)^k r^{\overline{k}}$. Dividing this equation be $k!$ proves the claim. □

Remark: Together with the equation $\sum_{k=0}^{n} \binom{m+k}{k} = \binom{m+n+1}{n}$ (see the remark concerning Pascal's triangle) that is valid for any $m \in \mathbb{C}$, we get

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = \sum_{k=0}^{m} \binom{k-n-1}{k} = \binom{m-n}{m} = (-1)^m \binom{n-1}{m}.$$

Hence, we have a formula for the alternating sum of a row in Pascal's triangle. Moreover, the equation has a combinatorial interpretation: both sides count the number of m-subsets of $\{1, \dots, n-1\}$ if m is even (or minus this number if m is odd).

1.5 Permutations

An n-permutation is a permutation $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$.

For a permutation π, a **cycle** is a vector $(i_1, i_2, \dots, i_t)$ such that $\pi(i_j) = i_{j+1}$ for $j \in \{1, \dots, t-1\}$ and $\pi(i_t) = i_1$. Hence a fixed pointed corresponds to a cycle of length 1.

Definition 4 *For $n, k \in \mathbb{N}$ let $s_{n,k}$ be the number of n-permutations with k cycles. The numbers $s_{n,k}$ are called* **Stirling numbers of the first kind**.

Proposition 13 *For $n, k \in \mathbb{N} \setminus \{0\}$ we have:*

$$s_{n,k} = s_{n-1,k-1} + (n-1)s_{n-1,k}.$$

Proof: There are $s_{n-1,k-1}$ n-permutations with k cycles where n is a fixed point and $(n-1)s_{n-1,k}$ n-permutations with k cycles where n is not a fixed point. $\square$

Stirling's triangle of the first kind

Definition 5 *For $n \in \mathbb{N}$ we define $H_n := \sum_{i=1}^{n} \frac{1}{i}$. We call H_n the n-th* **harmonic number**.

It is a well-know fact (that can easily be proved by using the fact that the derivative of $x \mapsto \ln x$ is $\frac{1}{x}$) that $H_n \in \Theta(\ln n)$, More precisely, we have for positive integers n:

$$\ln n + \frac{1}{n} \leq H_n \leq \ln n + 1.$$

Proposition 14 *For $n \in \mathbb{N} \setminus \{0\}$ we have $s_{n,1} = (n-1)! H_{n-1}$.*

Proof: Induction in n. For $n = 1$ we have 0 on both sides of the equation, so let $n > 1$. Then:

$$\frac{s_{n,2}}{(n-1)!} = \frac{s_{n-1,1}}{(n-1)!} + \frac{(n-1)s_{n-1,2}}{(n-1)!} = \frac{(n-2)!}{(n-1)!} + \frac{s_{n-1,2}}{(n-2)!} = \frac{1}{n-1} + H_{n-2} = H_{n-1}.$$

$\square$

There is a surprising connection between the two kinds of Stirling numbers: they are coefficients that allow to switch between the bases $\{r^0, r^1, r^2, \dots\}$ and $\{r^{\underline{0}}, r^{\underline{1}}, r^{\underline{2}}, \dots\}$ of the vector space of the polynomials. We know already the following equation for $n, r \in \mathbb{N}$

$$r^n = \sum_{k=0}^{n} S_{n,k} r^{\underline{k}}.$$

By the polynomial method we can generalize this result to $r \in \mathbb{C}$. For the other direction (i.e. for writing $r^{\underline{n}}$ as a linear function of the polynomials r^k, we can use the Stirling numbers of th first kind:

Proposition 15 *For $n \in N$ and $r \in \mathbb{C}$, w have:*

$$r^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} s_{n,k} r^k.$$

Proof: Induction in n. For $n = 0$, both sides of the equation equal 1. Hence assume $n > 0$. We generalize the definition of $s_{n,k}$ to negative values of k by setting $s_{n,k} = 0$ for $k < 0$ and get:

$$\begin{aligned}
r^{\underline{n}} &= (r-n+1) r^{\underline{n-1}} \\
&= (r-n+1) \sum_{k=0}^{n-1} (-1)^{n-1-k} s_{n-1,k} r^k \\
&= \sum_{k=0}^{n-1} (-1)^{n-1-k} s_{n-1,k} r^{k+1} + \sum_{k=0}^{n-1} (-1)^{n-k} (n-1) s_{n-1,k} r^k \\
&= \sum_{k=1}^{n} (-1)^{n-k} s_{n-1,k-1} r^k + \sum_{k=0}^{n} (-1)^{n-k} (n-1) s_{n-1,k} r^k \\
&= \sum_{k=0}^{n} (-1)^{n-k} (s_{n-1,k-1} r^k + (n-1) s_{n-1,k} r^k) \\
&= \sum_{k=0}^{n} (-1)^{n-k} s_{n,k} r^k
\end{aligned}$$

□

We categorize permutations not only be the number of cycles but also on the number of cycles of a certain length.

Notation: For a permutation π, let $b_i(\pi)$ the number of cycles of length i (for $i \in \{\ ,\ldots,n\}$), and let $b(\pi) := \sum_{i=1}^{n} p_i(\pi)$ be the total number of cycles. The **type of a permutation** π is the formal term $t(\pi) = 1^{b_2(\pi)} 2^{b_2(\pi)} \ldots n^{b_n(\pi)}$ where we skip the number i with $b_i(\pi) = 0$.

Observation: $\sum_{i=1}^{n} i b_i(\pi) = n$.

Proposition 16 *There are $\sum_{k=0}^{n} p_{n,k}$ types of n-permutations.*

Proof: For every type $1^{b_2(\pi)} 2^{b_2(\pi)} \ldots n^{b_n(\pi)}$ we find a representation of n as a sum of $b(\pi)$ positive integers where the number i occurs $p_i(\pi)$ times as a summand. This leads to a bijection between the set of types of n-permutations and the set of partitions of n. □

Proposition 17 *There are*

$$\frac{n!}{b_1! \dots b_n! 1^{b_1} 2^{b_2} \dots n^{b_n}}$$

n-permutations of type $1^{b_1} 2^{b_2} \dots n^{b_n}$.

Proof: First consider empty cycles:

$$\underbrace{(\cdot)(\cdot)\dots(\cdot)}_{b_1(\pi) \text{ times}}\underbrace{(\cdot\cdot)(\cdot\cdot)\dots(\cdot\cdot)}_{b_2(\pi) \text{ times}}\underbrace{(\cdot\cdot\cdot)(\cdot\cdot\cdot)\dots(\cdot\cdot\cdot)}_{b_3(\pi) \text{ times}}\dots$$

There are $n!$ ways to fill these cycles with numbers from $\{1, \dots, n\}$ (without repeating numbers). Moreover, each such assignment yields an n-permutation of type $1^{b_1} 2^{b_2} \dots n^{b_n}$. However, for each $i \in \{1, \dots, n\}$, all assignments that differ just in the order of th b_i cycles lead to the same permutation. And for each cycle of length i there are i assignments giving the same permutation because there are i ways to choose the first element in the cycle. Hence, for each n-permutation π, there are $b_1! \dots b_n! 1^{b_1} 2^{b_2} \dots n^{b_n}$ assignments that encode π. □

In particular we have

$$s_{n,k} = \sum_{(b_1, \dots, b_n)} \frac{n!}{b_1! \dots b_n! 1^{b_1} \dots n^{b_n}}$$

where we sum over all vectors $(b_1, \dots, b_n)$ with $\sum_{i=1}^n i b_i = n$ and $\sum_{i=1}^n b_i = k$.

Similarly:

$$n! = \sum_{(b_1, \dots, b_n)} \frac{n!}{b_1! \dots b_n! 1^{b_1} \dots n^{b_n}}$$

where we sum over all vectors $(b_1, \dots, b_n)$ with $\sum_{i=1}^n i b_i = n$.

1.6 Further Combinatorial Techniques

Inclusion-exclusion principle:

Proposition 18 *(Inclusion-exclusion principle) Let $A_1, \dots, A_n$ be finite sets. Then:*

$$|\bigcup_{i=1}^n A_i| = \sum_{r=1}^n (-1)^{r-1} \sum_{1 \le i_1 < \dots < i_r \le n} |\bigcap_{j=1}^r A_{i_j}|.$$

Proof: The statement is trivial if all sets A_i are empty, so let $a \in \bigcup_{i=1}^n A_i$. We have to show that a is counted on the right-hand side of the equation exactly once. Let $k := |\{i \in \{1, \dots, n\} \mid$

$a \in A_i\}|$. Then a is counted $\binom{k}{r}$ times in the sum $\sum_{1 \le i_1 < \cdots < i_r \le n} |\bigcap_{j=1}^r A_{i_j}|$ (for $r \in \{1, \ldots, n\}$). Hence in total, a is counted $\sum_{r=1}^k (-1)^{r-1}\binom{k}{r}$. By the binomial theorem, we have

$$0 = (-1+1)^k = \sum_{r=0}^k \binom{k}{r}(-1)^r 1^{k-r} = 1 - \sum_{r=1}^k (-1)^{r-1}\binom{k}{r}.$$

Thus, $\sum_{r=1}^k (-1)^{r-1}\binom{k}{r} = 1$, which means that a is counted exactly once in the sum on the right-hand side. □

Proposition 19 *The number of fixed-point free n-permutations is $D_n := n! \sum_{r=0}^n \frac{(-1)^r}{r!}$.*

Proof: Let A_i be the set of all n-permutations π with $\pi(i) = i$. Thus, the number of fixed-point free n-permutations is

$$n! - |\bigcup_{i=1}^n A_i| = n! - \sum_{r=1}^n (-1)^{r-1} \sum_{1 \le i_1 < \cdots < i_r \le n} |\bigcap_{j=1}^r A_{i_j}|.$$

For and $1 \le i_1 < \cdots < i_r \le n$ we obviously have $|\bigcap_{j=1}^r A_{i_j}| = (n-r)!$. Moreover, there are $\binom{n}{r}$ ways to choose integers $i_1, \ldots, i_r$ with $1 \le i_1 < \cdots < i_r \le n$ (for $r \in \{1, \ldots, n\}$). Thus the number of fixed-point free n-permutations is

$$\begin{aligned} n! - \sum_{r=1}^n \binom{n}{r}(-1)^{r-1}(n-r)! &= \sum_{r=0}^n \binom{n}{r}(-1)^r(n-r)! \\ &= \sum_{r=0}^n \frac{n!}{r!(n-r)!}(-1)^r(n-r)! \\ &= n! \sum_{r=0}^n \frac{(-1)^r}{r!} \end{aligned}$$

□

The numbers D_n are called the **derangement numbers**. We have $\lim_{n\to\infty} \sum_{r=0}^n \frac{(-1)^r}{r!} = \frac{1}{e}$, so for large n the fraction of the fixed-point free n-permutations among all n-permutations is approximately $\frac{1}{e}$.

Another application of the inclusion-exclusion principle:

Proposition 20 *For any n the number of ways to write n as a sum of odd natural numbers is the number of ways to write n as a sum of different positive integers.*

Proof: For $n \in \mathbb{N} \setminus \{0\}$ let $p(n) = \sum_{k=1}^n P_{n,k}$ be the number of ways to partition n. Then, for numbers $1 \le i_1 < \cdots < i_r \le n$, the number of partitions where these numbers occur twice is $p(n - 2\sum_{j=1}^{r} i_j)$. However, this is also the number of partitions where the even numbers $2i_i, \ldots, 2i_r$ occur. Hence, both for the number of partitions in odd summands and for the number of partitions in different summands we get:

$$p(n) - \sum_{r=1}^{n} (-1)^{r-1} \sum_{1 \le i_1 < \cdots < i_r \le n} p\left(n - 2\sum_{j=1}^{r} i_j\right).$$

$\square$

Doubly counting:

The **principle of doubly counting** is based on the following simple observation for a relation $R \subseteq S \times T$ we can count the pairs in R in two different ways and get:

$$\sum_{s \in S} |\{t \in T \mid (s,t) \in R\}| = \sum_{t \in T} |\{s \in S \mid (s,t) \in R\}|$$

Examples:

- We want to compute the average number of divisors of the integers in $\{1, \ldots, n\}$. To this end, we set $S = T = \{1, \ldots, n\}$ and define the relation $R \subseteq S \times T$ by $(s,t) \in R :\Leftrightarrow t|s$. Then, the average number of divisors is

$$\begin{aligned} \frac{1}{n} \sum_{s \in S} |\{t \in \{1, \ldots, n\} : t|s\}| &= \frac{1}{n} \sum_{s \in S} |\{t \in T : (s,t) \in R\}| \\ &= \frac{1}{n} \sum_{t \in T} |\{s \in S : (s,t) \in R\}| \\ &= \frac{1}{n} \sum_{t \in T} |\{s \in \{1, \ldots, n\} : t|s\}| \\ &= \frac{1}{n} \sum_{t \in T} \lfloor \frac{n}{t} \rfloor \quad \in \quad \left[\left(\sum_{i=1}^{n} \frac{1}{i} \right) - \frac{1}{n}, \sum_{i=1}^{n} \frac{1}{i} \right] \end{aligned}$$

  Hence, the average number of divisors of the number in $\{1, \ldots, n\}$ is approximately $H_n = \sum_{i=1}^{n} \frac{1}{i}$ which is $\Theta(\ln n)$.

- Let G be an undirected graph. For $S = V(G)$ and $T = E(G)$ we define the relation $R \subseteq V(G) \times E(G)$ by $(v,e) \in R :\Leftrightarrow v \in e$. Then doubly counting proves $\sum_{e \in E(G)} |\{v \in V(G) \mid v \in e\}| = \sum_{v \in V} |\{e \in E(G) \mid v \in e\}|$, so $2|E(G)| = \sum_{v \in V(G)} |\delta_G(v)|$.

- Claim: An undirected graph G that does not contain a cycle of length 4 has at most $\lfloor \frac{n}{4}(1 + \sqrt{4n-3}) \rfloor$ edges (where $n = |V(G)|$).
  Proof of the claim: For $S = V(G)$ and $T = \{\{v,w\} \subseteq V(G) \mid v \neq w\}$ we define the relation $R \subseteq S \times T$ by

$$(u, \{v,w\}) \in R :\Leftrightarrow \{u,v\} \in E(G) \text{ and } \{u,w\} \in E(G).$$

Then, doubly counting leads to

$$\sum_{s \in S} |\{t \in T \mid (s,t) \in R\}| = \sum_{u \in V(G)} \binom{|\delta_G(u)|}{2} = \sum_{t \in T} |\{s \in S \mid (s,t) \in R\}| \le |T| = \binom{n}{2}$$

where the inequality follows from the fact that each two nodes v, w can have at most one common neighbour (otherwise we had a cycle of length 4 in G). Thus,

$$\sum_{u \in V(G)} |\delta_G(u)|^2 \le n(n-1) + \sum_{u \in V(G)} |\delta_G(u)|$$

Moreover, with $\alpha(u) := \frac{2m}{n} - |\delta_G(u)|$ for $u \in V(G)$ (where $m = |E(G)|$) we get

$$\sum_{u \in V(G)} |\delta(u)|^2 = \sum_{u \in V(G)} \left(\frac{2m}{n} - \alpha(v)\right)^2 = \sum_{u \in V(G)} \left(\frac{2m}{n}\right)^2 - \frac{4m}{n} \sum_{u \in V(G)} \alpha(u) + \sum_{u \in V(G)} \alpha(u)^2 \ge \frac{4m^2}{n}$$

because $\sum_{u \in V(G)} \alpha(u) = 0$. Hence, $\frac{4m^2}{n} \le n(n-1) + 2m$, which proves the claim.

(Generalized) pigeon hole principle:

For an n-set N and an r-set R with $n > r$ and a mapping $f : N \to R$ there is an $a \in R$ with $|f^{-1}(a)| \ge \lfloor \frac{n-1}{r} \rfloor + 1$. In particular there must be an $a \in R$ with $|f^{-1}(a)| \ge 2$.

This principle is obviously true because otherwise we had $n = \sum_{a \in R} |f^{-1}(a)| \le r \lfloor \frac{n-1}{r} \rfloor < n$.

Simple Applications: Let $a_1, \dots, a_n$ (with $n \ge 1$) be a finite sequence of positive integers.

- **Claim:** There are numbers $k, l \in \{1, \dots, n\}$ such that $\sum_{i=k}^{l} a_i$ is an integral multiple of n.
  Proof: Let $N = \{\sum_{i=1}^{l} a_i \mid l \in \{0, \dots, n\}\}$ and $R = \{0, \dots, n-1\}$. We define $f(m)$ to be the remainder of m on division by n. Since $|N| = n+1 > n = |R|$, there are numbers m, l with $m < l$ and $f(\sum_{i=1}^{m} a_i) = f(\sum_{i=1}^{l} a_i)$. Thus, $\sum_{i=m+1}^{l} a_i$ is an integral multiple of n.

- **Claim:** If $\{a_1, \dots, a_n\} \subseteq \{1, \dots, 2n-2\}$, then there are numbers $i, j \in \{1, \dots, n\}$ with $i \ne j$ such that a_i is an integral multiple of a_j.
  Proof: Let $N = \{a_1, \dots, a_n\}$ and $R = \{2i - 1 \mid i \in \{1, \dots, n-1\}$. We can write each number a_i in a unique way as $a_i = 2^{k_i} b_i$ where b_i is an odd number. We define $f : N \to R$ by $f(a_i) = b_i$. Then, by the pigeon-hole principle, there are numbers a_i and a_j with $2^{k_i} b_i = 2^{k_j} b_j$ with $i \ne j$ and $k_i \le k_j$, so a_j is an integral multiple of a_i.

- **Claim:** If the numbers a_i are pairwise different and $n > k \cdot l$ for some positive integers l, k, then there is an increasing subsequence $a_{i_1} < a_{i_2} < \dots < a_{i_{k+1}}$ $(i_1 < i_2 < \dots < i_{k+1})$ of length $k+1$ or a decreasing subsequence $a_{i_1} > a_{i_2} > \dots > a_{i_{l+1}}$ $(i_1 < i_2 < \dots < i_{l+1})$ of length $l+1$.
  Proof: Assume that there is no increasing subsequence of length $k+1$. For $i \in \{1, \dots, n\}$ we define t_i to be the length of the longest increasing subsequence starting with a_i. Hence,

by setting $f(a_i) = t_i$, we get a mapping $f : \{a_1, \ldots, a_n\} \to \{1, \ldots, k\}$. Thus there must be an $j \in \{1, \ldots, k\}$ with $|f^{-1}(j)| > l$. Obviously, the elements of $f^{-1}(j)$ form a decreasing subsequence.

Theorem 21 *(Ramsey's Theorem): For $k, l \in \mathbb{N} \setminus \{0\}$, there is a smallest number $R(k, l)$ such that any graph with at least $R(k, l)$ nodes contains a clique of size k or a stable set of size l.*

Proof: Induction on $k + l$. If $k = 1$ or $l = 1$, then obviously $R(k, l) = 1$. Hence, we assume $k > 1$ and $l > 1$.

Claim: $R(k, l) \leq R(k - 1, l) + R(k, l - 1)$.

Proof of the claim: Let G be a graph with $R(k-1, l) + R(k, l-1)$ nodes, and let $v \in V(G)$ be a node of G. Let X be the set of neighbours of v in G and $Y := V(G) \setminus X$. Thus, $|X| \geq R(k-1, l)$ or $|Y| \geq R(k, l-1)$. If $|X| \geq R(k-1, l)$, then X contains a stable set of size l (so we are done) or a clique of size $k - 1$. Since all elements of X are neighbours of v, such a clique can be extended by adding v to a clique of size k, so, again, we a re done. The case $|Y| \geq R(k, l - 1)$ can be handled analogously. □

The number $R(k, l)$ are called **Ramsey numbers**.

In particular for $m > 1$: $R(2, m) = R(m, 2) = m$.

Theorem 22 *For $k, l \in \mathbb{N} \setminus \{0\}$ we have $R(k, l) \leq \binom{k+l-2}{k-1}$.*

Proof: For $k = 1$, we have $R(1, l) \leq \binom{l-1}{0} = 1$. For $l = 1$, we have $R(k, l) \leq \binom{k-1}{k-1} = 1$. In general, we get:

$$R(k, l) \leq R(k-1, l) + R(k, l-1) \leq \binom{k+l-3}{k-2} + \binom{k+l-3}{k-1} = \binom{k+l-2}{k-1}.$$

□

2 Computation of Sums

2.1 Sums: Direct Methods

Induction:

For a non-trivial example of induction we refer to Aigner [2007], p. 42.

Index transformation:

Make use of the observation that sums can be computed in many different ways:

$$\sum_{k=m}^{n} a_k = \sum_{k=m+i}^{n+i} a_{k-i} = \sum_{k=m-i}^{n-i} a_{k+i} = \sum_{k=0}^{n-m} a_{m+k} = \sum_{k=0}^{n-m} a_{n-k}$$

For example, assume that we want to compute $S_n = \sum_{k=0}^{n} ka$. Then, $S_n = \sum_{k=0}^{n}(n-k)a$, so $2S_n = \sum_{k=0}^{n} ka + \sum_{k=0}^{n}(n-k)a = \sum_{k=0}^{n} na = (n+1)na$, which implies $S_n = \frac{1}{2}(n+1)na$.

Isolating terms:

For a sum $S_n = \sum_{k=0}^{n} a_k$ isolate the first and the last terms from S_{n+1}: $S_{n+1} = S_n + a_{n+1} = a_0 + \sum_{k=1} n + 1a_k = a_0 + \sum_{k=0} na_{k+1}$

Examples:

- Consider the (finite) geometric sum $S_n = \sum_{k=0}^{n} a^k$. We get

  $$S_{n+1} = S_n + a^{n+1} = 1 + \sum_{k=0}^{n} a^{k+1} = 1 + a\sum_{k=0}^{n} a^k = 1 + aS_n.$$

  Thus, $S_n + a^{n+1} = 1 + aS_n$, so $S_n = \frac{a^{n+1}-1}{a-1}$ (if $a \neq 1$).

- Let $S_n = \sum_{k=0}^{n} k2^k$. Then,

  $$S_{n+1} = S_n + (n+1)2^{n+1} = 0 + \sum_{k=0}^{n}(k+1)2^{k+1} = 2\sum_{k=0}^{n} k2^k + 2\sum_{k=0}^{n} 2^k = 2S_n + 2^{n+2} - 2,$$

  so $S_n = (n-1)2^{n+1} + 2$.

2.2 Difference and Sum Operators

Definition 6 *For $a \in \mathbb{Z}$ the* **translation operator** $E^a : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ *maps $f \in \mathbb{R}^{\mathbb{Z}}$ to $E^a f$ where $E^a f(x) = f(x+a)$ for all $x \in \mathbb{R}$.*

Hence $I := E^0$ is the identity.

For two operators, $P, Q : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$, we can define their sum by $(P+Q)f = Pf + Qf$ and a multiplication by a scalar α by setting $(\alpha P)f = \alpha(Pf)$. We denote their composition by QP, so $(QP)f = Q(Pf)$.

Moreover, we define $\Delta : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ as $\Delta = E^1 - I$, so for $f : \mathbb{Z} \to \mathbb{R}$ we have $\Delta f(x) = f(x+1) - f(x)$. This is the **forward difference operator**. Similarly, we define $\nabla : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$

as $\nabla = I - E^{-1}$, so for $f : \mathbb{Z} \to \mathbb{R}$ we have $\nabla f(x) = f(x) - f(x-1)$. This is the **backward difference operator**.

Examples: For $n \in \mathbb{N}$:

$$\Delta x^{\underline{n}} = (x+1)^{\underline{n}} - x^{\underline{n}} = (x+1)x^{\underline{n-1}} - (x-n+1)x^{\underline{n-1}} = nx^{\underline{n-1}}$$

and

$$\nabla x^{\overline{n}} = x^{\overline{n}} - (x-1)^{\overline{n}} = (x+n-1)x^{\overline{n-1}} - (x-1)x^{\overline{n-1}} = nx^{\overline{n-1}}$$

We generalize the falling and rising factorials to negative exponents by setting (for $n \in \mathbb{N} \setminus \{0\}$):

$$x^{\underline{-n}} := \frac{1}{(x+1)\dots(x+n)}.$$

and

$$x^{\overline{-n}} := \frac{1}{(x-1)\dots(x-n)}.$$

Theorem 23 *For $n \in \mathbb{Z}$, we have:*

$$\Delta x^{\underline{n}} = nx^{\underline{n-1}}$$

and

$$\nabla x^{\overline{n}} = nx^{\overline{n-1}}$$

Proof: For $n \geq 0$ we have already proved the statement. The rest follows from

$$\Delta x^{\underline{-n}} = (x+1)^{\underline{-n}} - x^{\underline{-n}} = (x+1)x^{\underline{-n-1}} - (x+n+1)x^{\underline{-n-1}} = -nx^{\underline{-n-1}}$$

and

$$\nabla x^{\overline{-n}} = (x)^{\overline{-n}} - (x-1)^{\overline{-n}} = (x-n-1)x^{\overline{-n-1}} - (x+1)x^{\overline{-n-1}} = -nx^{\overline{-n-1}}$$

□

Definition 7 *For two mappings $f, g : \mathbb{Z} \to \mathbb{R}$, we call f a* **(discrete) antiderivative** *of g, if $\Delta f = g$. We write $f = \sum g$ and call f an* **indefinite sum***.*

Thus:

$$\Delta f = g \Leftrightarrow f = \sum g.$$

Theorem 24 *If f is an antiderivative of g then for all $a, b \in \mathbb{Z}$ with $a < b$:*

$$\sum_{k=a}^{b} g(k) = f(b+1) - f(a).$$

Proof: Since $\Delta f = g$ we have $f(k+1) - f(k) = g(k)$ for all $k \in \mathbb{Z}$. Thus,

$$\sum_{k=a}^{b} g(k) = \sum_{k=a}^{b} (f(k+1) - f(k)) = f(b+1) - f(a).$$

$\square$

Notation: For $f = \sum g$ and $a, b \in \mathbb{Z}$ with $a < b$, we write $\sum_a^{b+1} g(x) := f(x)|_a^{b+1} := f(b+1) - f(a)$. Hence:

$$\sum_{a}^{b+1} g(x) = \sum_{k=a}^{b} g(k).$$

Of course, $\sum g$ is defined only up to an additional constant. Nevertheless, we will use $\sum$ like an operator $\mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$.

Observation: Both Δ and $\sum$ are linear operators.

Since $\Delta x^{\underline{n+1}} = (n+1)x^{\underline{n}}$, we have $\sum x^{\underline{n}} = \frac{x^{\underline{n+1}}}{(n+1)}$ if $n \neq -1$. It remains to find $\sum x^{\underline{-1}}$. A function f with $f = \sum x^{\underline{-1}}$ satisfies $\frac{1}{x+1} = x^{\underline{-1}} = \Delta f = f(x+1) - f(x)$. Thus $H_x = \sum_{i=1}^{x} \frac{1}{i}$ is an antiderivative of $x^{\underline{-1}}$. Hence, we get:

$$\sum x^{\underline{n}} = \begin{cases} \frac{x^{\underline{n+1}}}{n+1} & n \neq -1 \\ H_x & n = -1 \end{cases}$$

Application: Compute $\sum_{k=0}^{n} k^2$: We have $x^2 = x(x-1) + x = x^{\underline{2}} + x^{\underline{1}}$, so

$$\begin{aligned} \sum_{k=0}^{n} k^2 &= \sum_{0}^{n+1} x^2 = \sum_{0}^{n+1} x^{\underline{2}} + \sum_{0}^{n+1} x^{\underline{1}} = \frac{x^{\underline{3}}}{3} \Big|_0^{n+1} + \frac{x^{\underline{2}}}{2} \Big|_0^{n+1} \\ &= \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2} = \frac{(n+1)n(n-1)}{3} + \frac{(n+1)n}{2} = \frac{n(n+\frac{1}{2})(n+1)}{3} \end{aligned}$$

More general: $x^m = \sum_{k=0}^{m} S_{m,k} x^{\underline{k}}$. Therefore:

$$\sum_{k=0}^{n} k^m = \sum_{0}^{n+1} x^m = \sum_{0}^{n+1} \left(\sum_{k=0}^{m} S_{m,k} x^{\underline{k}} \right) = \sum_{k=0}^{m} S_{m,k} \sum_{0}^{n+1} x^{\underline{k}} = \sum_{k=0}^{m} S_{m,k} \frac{x^{\underline{k+1}}}{k+1} \Big|_0^{n+1} = \sum_{k=0}^{m} S_{m,k} \frac{(n+1)^{\underline{k+1}}}{k+1}.$$

Further examples:

- For $c \in \mathbb{R}$ we have $\Delta c^x = c^{x+1} - c^x = (c-1)c^x$. Hence, for $c \neq 1$: $\sum c^x = \frac{c^x}{c-1}$. In particular, for $c = 2$ we get $\Delta 2^x = 2^x$ and $\sum 2^x = 2^x$.
- For $x \in \mathbb{R}$ and $m \in \mathbb{Z}$ we have $\binom{x+1}{m+1} = \binom{x}{m} + \binom{x}{m+1}$. Thus $\Delta \binom{x}{m+1} = \binom{x}{m}$ and $\sum \binom{x}{m} = \binom{x}{m+1}$.

For a function $f : \mathbb{Z} \to \mathbb{R}$, we have

$$\Delta^n f(x) = (E - I)^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} E^k f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+k)$$

In particular, for $x = 0$:

$$\Delta^n f(0) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k) \tag{2}$$

Theorem 25 *(Newton representation of polynomials): For a polynomial f of degree n, we have*

$$f(x) = \sum_{k=0}^{n} \frac{\Delta^k f(0)}{k!} x^{\underline{k}}$$

Proof: Since the polynomials $x^{\underline{k}}$ are a basis of the space of polynomials, we can write f in a unique way as $f(x) = \sum_{k=0}^{n} b_k x^{\underline{k}}$. It remains to show that $b_k = \frac{\Delta^k f(0)}{k!}$ (for $\{k \in 0 \ldots, n\}$).

We have $\Delta^k x^{\underline{i}} = i(i-1)\ldots(i-k+1)x^{\underline{i-k}} = i^{\underline{k}} x^{\underline{i-k}}$. Hence:

$$\Delta^k f(x) = \Delta^k \sum_{i=0}^{n} b_i x^{\underline{i}} = \sum_{i=0}^{n} b_i i^{\underline{k}} x^{\underline{i-k}}$$

Thus $\Delta^k f(0) = \sum_{i=0}^{n} b_i i^{\underline{k}} 0^{\underline{i-k}} = b_k k^{\underline{k}}$ because $i^{\underline{k}} = 0$ for $i < k$ and $0^{\underline{i-k}} = 0$ for $i > k$. Since $k^{\underline{k}} = k!$, this proves the theorem. $\square$

Corollary 26 *For $n, k \in \mathbb{N}$, we have:*

$$S_{n,k} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n$$

Proof: We know that $x^n = \sum_{k=0}^{n} S_{n,k} x^{\underline{k}}$, so by the previous theorem we have, with $f(x) = x^n$:

$$S_{n,k} = \frac{\Delta^k f(0)}{k!} = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n.$$

For the last equation, we applied (2). $\square$

Theorem 27 *(Partial summation) For functions $u, v : \mathbb{Z} \to \mathbb{R}$, we have:*

$$\sum (u \Delta v) = uv - \sum ((Ev)\Delta u)$$

Proof: We have

$$\begin{aligned}\Delta uv(x) &= u(x+1)v(x+1) - u(x)v(x)\\ &= u(x)(v(x+1)-v(x)) + v(x+1)(u(x+1)-u(x))\\ &= u(x)\Delta v(x) + Ev(x)\Delta u(x),\end{aligned}$$

so $\Delta uv = u\Delta v + (Ev)\Delta u$. Applying the operator $\sum$ to this equation proves the statement of the theorem. $\square$

Applications:

- Compute $\sum\limits_{k=0}^{n} k2^k$.
  Thus, we want to compute $\sum x2^x$. We apply the theorem with $u(x) = x$ and $\Delta v(x) = 2^x$, so $\Delta u(x) = 1$ and $v(x) = 2^x$. This leads to:

$$\begin{aligned}\sum_{k=0}^{n} k2^k &= \sum_{0}^{n+1} x2^x = x2^x|_0^{n+1} - \sum_{0}^{n+1} 2^{x+1} = x2^x|_0^{n+1} - 2\cdot 2^x|_0^{n+1}\\ &= (n+1)2^{n+1} - 2\cdot 2^{n+1} + 2 = (n-1)2^{n+1} + 2\end{aligned}$$

- Compute $\sum\limits_{k=0}^{n} H_k$.
  Apply the theorem with $u(x) = H_x$ and $\Delta v(x) = 1$, so $\Delta u(x) = \frac{1}{1+x}$ and $v(x) = x$. This leads to:

$$\begin{aligned}\sum_{k=1}^{n} H_k &= \sum_{1}^{n+1} H_x x^{\underline{0}} = H_x x|_1^{n+1} - \sum_{1}^{n+1}(x+1)\frac{1}{1+x} = H_{n+1}(n+1) - 1 - x|_1^{n+1}\\ &= H_{n+1}(n+1) - 1 - (n+1) + 1 = (n+1)(H_{n+1} - 1)\end{aligned}$$

- Compute $\sum\limits_{k=1}^{n} \binom{k}{m} H_k$ (for $m \in \mathbb{N}\setminus\{0\}$).
  Apply the theorem with $u(x) = H_x$ and $\Delta v(x) = \binom{x}{m}$, so $\Delta u(x) = \frac{1}{1+x}$ and $v(x) = \binom{x}{m+1}$. This leads to:

$$\begin{aligned}\sum_{k=1}^{n}\binom{k}{m}H_k &= \sum_{1}^{n+1}\binom{x}{m}H_x = H_x\binom{x}{m+1}\Big|_1^{n+1} - \sum_{1}^{n+1}\binom{x+1}{m+1}\frac{1}{1+x}\\ &= H_{n+1}\binom{n+1}{m+1} - H_1\binom{1}{m+1} - \frac{1}{m+1}\sum_{1}^{n+1}\binom{x}{m}\\ &= H_{n+1}\binom{n+1}{m+1} - \frac{1}{m+1}\binom{x}{m+1}\Big|_1^{n+1}\\ &= H_{n+1}\binom{n+1}{m+1} - \frac{1}{m+1}\left(\binom{n+1}{m+1} - \binom{1}{m+1}\right)\\ &= \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right)\end{aligned}$$

2.3 Inversions

Definition 8 *A* **basis sequence** *is a sequence of polynomials* $(p_i)_{i\in\mathbb{N}} = p_0(x), p_1(x), \ldots$ *where* p_i *is a polynomial of degree* i *(for* $i \in \mathbb{N}$*).*

Examples of basis sequences are $(x^n)_{n\in\mathbb{N}}$ and $(x^{\underline{n}})_{n\in\mathbb{N}}$.

It is easy to check that if $p_0(x), p_1(x), \ldots$ is a basis sequence and $p(x)$ is a polynomial degree n then there are unique numbers $a_0, \ldots, a_n$ such that $p(x) = \sum_{k=0}^{n} a_k p_k(x)$.

Now let $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ be two basis sequences. Then there unique numbers $a_{n,k}$ and $b_{n,k}$ for $k \leq n$ with

$$q_n(x) = \sum_{k=0}^{n} a_{n,k} p_k(x)$$

and

$$p_n(x) = \sum_{k=0}^{n} b_{n,k} q_k(x)$$

For $k > n$ we set all numbers $a_{n,k}$ and $b_{n,k}$ to 0. The numbers $a_{n,k}$ and $b_{n,k}$ are called **connection coefficients**.

We have

$$q_n(x) = \sum_{k=0}^{n} a_{n,k} p_k(x) = \sum_{k=0}^{n} a_{n,k} \sum_{m=0}^{k} b_{k,m} q_m(x) = \sum_{m=0}^{n} q_m(x) \sum_{k=m}^{n} a_{n,k} b_{k,m}$$

Hence

$$\sum_{k=m}^{n} a_{n,k} b_{k,m} = \sum_{k=1}^{n} a_{n,k} b_{k,m} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Therefore, if $(n \in \mathbb{N})$ $A_n = (a_{i,j})_{1\leq,i,j\leq n}$ and $B_n = (b_{i,j})_{1\leq,i,j\leq n}$, then $A_n B_n = I_n$ where I_n is the $n \times n$-identity matrix.

Theorem 28 *Let* $(p_n)_{n\in\mathbb{N}}$ *and* $(q_n)_{n\in\mathbb{N}}$ *be two basis sequences with connection coefficients* $a_{n,k}$ *and* $b_{n,k}$*. Then for two sequences* $(u_n)_{n\in\mathbb{N}}$ *and* $(v_n)_{n\in\mathbb{N}}$ *the following statements are equivalent:*

$$\forall n \in \mathbb{N} \quad v_n = \sum_{k=0}^{n} a_{n,k} u_k$$

and

$$\forall n \in \mathbb{N} \quad u_n = \sum_{k=0}^{n} b_{n,k} v_k$$

Proof: Let $n \in \mathbb{N}$. The matrices A_n and B_n are inverse to each other, so for any two vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$, we have

$$v = A_n u \quad \Leftrightarrow u = B_n v$$

$\square$

Examples:

- Stirling numbers:
  Consider the basis sequences $(x^n)_{n\in\mathbb{N}}$ and $(x^{\underline{n}})_{n\in\mathbb{N}}$. We know from previous results that

  $$x^n = \sum_{k=0}^{n} S_{n,k} x^{\underline{k}}.$$

  and

  $$x^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} s_{n,k} x^k.$$

  Thus the numbers $S_{n,k}$ and $(-1)^{n-k} s_{n,k}$ are the connection coefficients of the basis sequences $(x^{\underline{n}})_{n\in\mathbb{N}}$ and $(x^n)_{n\in\mathbb{N}}$. This gives us

  $$\sum_{k\geq 0} S_{n,k}(-1)^{k-m} s_{k,m} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

  Moreover, for any sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ we have

  $$\left(\forall n \in \mathbb{N} \quad v_n = \sum_{k=0}^{n} S_{n,k} u_k\right) \quad \Leftrightarrow \quad \left(\forall n \in \mathbb{N} \quad u_n = \sum_{k=0}^{n} (-1)^{n-k} s_{n,k} v_k\right)$$

  This equivalence is called **Stirling inversion**.

- Binomial coefficients
  For $n \in \mathbb{N}$, we have:

  $$x^n = ((x-1)+1)^n = \sum_{k=0}^{n} \binom{n}{k} (x-1)^k$$

  and

  $$(x-1)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x^k$$

  Therefore, the numbers $\binom{n}{k}$ and $(-1)^{n-k}\binom{n}{k}$ are the connections coefficients of the basis sequences $((x-1)^n)_{n\in\mathbb{N}}$ and $((x)^n)_{n\in\mathbb{N}}$.

  This implies

  $$\sum_{k\geq 0} \binom{n}{k} (-1)^{k-m} \binom{k}{m} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Moreover, for any sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ we have

$$\left(\forall n\in\mathbb{N}\quad v_n=\sum_{k=0}^{n}\binom{n}{k}u_k\right)\quad\Leftrightarrow\quad\left(\forall n\in\mathbb{N}\quad u_n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}v_k\right)$$

This equivalence is called **binomial inversion** By replacing u_n by $(-1)^n u_n$ we get a more symmetric version:

For any sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ we have

$$\left(\forall n\in\mathbb{N}\quad v_n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}u_k\right)\quad\Leftrightarrow\quad\left(\forall n\in\mathbb{N}\quad u_n=\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}v_k\right)$$

Application: We consider again the derangement numbers D_n. We have

$$n!=\sum_{k=0}^{n}\binom{n}{k}D_k$$

because $\binom{n}{n-k}D_k=\binom{n}{k}D_k$ is the number of n-permutations with exactly $n-k$ fixed points. By applying the first version of the binomial inversion (with $v_n=n!$ and $u_k=D_k$), we get

$$D_n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k!=n!\sum_{k=0}^{n}\frac{(-1)^{n-k}}{(n-k)!}=n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$$

3 Linear Recursions of Depth 1

Theorem 29 *Let $T_0, T_1, \ldots$ be a sequence that is given by numbers α and a_n, b_n with $a_n\neq 0$ $(n\in\mathbb{N}\setminus\{0\})$ and the following recursion:*

- $T_0=\alpha$
- $T_n=a_nT_{n-1}+b_n$

Then, for all $n\in\mathbb{N}$:

$$T_n=\prod_{k=1}^{n}a_k\left(\sum_{k=1}^{n}\frac{b_k}{\prod_{i=1}^{k}a_i}+T_0\right).$$

Proof: Induction in n. For $n=0$ the statement obviously holds, so assume $n>1$. Then

$$\begin{aligned}T_n &= a_nT_{n-1}+b_n=a_n\prod_{k=1}^{n-1}a_k\left(\sum_{k=1}^{n-1}\frac{b_k}{\prod_{i=1}^{k}a_i}+T_0\right)+b_n\\ &= \prod_{k=1}^{n}a_k\left(\sum_{k=1}^{n-1}\frac{b_k}{\prod_{i=1}^{k}a_i}+T_0\right)+b_n=\prod_{k=1}^{n}a_k\left(\sum_{k=1}^{n}\frac{b_k}{\prod_{i=1}^{k}a_i}+T_0\right).\end{aligned}$$

$\square$

Remark: The recursion in the theorem above is a linear inhomogeneous recursion (where the term "inhomogeneous" refers to the fact that the numbers b_n may be non-zero).

Corollary 30 *Let $T_0, T_1, \ldots$ be a sequence that is given by numbers a, b and α with $a \neq 1$ and the following recursion:*

- $T_0 = \alpha$
- $T_n = aT_{n-1} + b$

Then, for all $n \in \mathbb{N}$:

$$T_n = a^n T_0 + b\frac{a^n - 1}{a - 1}.$$

Proof: For $a = 0$, we have $T_0 = \alpha$ and $T_0 = b$ for all $n \in \mathbb{N} \setminus \{0\}$, so the statement holds. For $a \neq 0$, the statement follows from the previous theorem and the fact that $\sum_{k=1}^{n} \frac{1}{a^k} = -1 + \sum_{k=0}^{n} \frac{1}{a^k} = -1 + \frac{\left(\frac{1}{a}\right)^n - a}{1-a} = \frac{1-\left(\frac{1}{a}\right)^n}{a-1}$. $\square$

As an application, we again consider the derangement numbers. We have

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

because the set of fixed-point free n-permutations can be decomposed into $(n-1)D_{n-2}$ permutations where 1 is contained in a cycle of length 2 and $(n-1)D_{n-1}$ permutations where 1 is contained in a cycle of length at least 3.

Thus

$$D_n - nD_{n-1} = -(D_{n-1} - (n-1)D_{n-2}) = D_{n-2} - (n-2)D_{n-3}$$

and so on. Hence

$$D_n - nD_{n-1} = (-1)^{n-1}(D_1 - D_0) = (-1)^n.$$

Therefore, for $n \in \mathbb{N} \setminus \{0\}$:

$$D_n = nD_{n-1} + (-1)^n$$

This leads once again to

$$D_n = n! \left(\sum_{k=1}^{n} \frac{(-1)^k}{k!} + 1 \right) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

4 Generating Functions

Definition 9 *A* **generating function** *of a sequence* $(a_n)_{n\in\mathbb{N}}$ *is the formal expression* $\sum_{n\geq 0} a_n z^n$.

Let $\sum_{n\geq 0} a_n z^n$ and $\sum_{n\geq 0} b_n z^n$ be two generating functions. Then, their sum is $\sum_{n\geq 0}(a_n + b_n)z^n$, and (for $c \in \mathbb{R}$) $c\sum_{n\geq 0} a_n z^n$ is $\sum_{n\geq 0}(ca_n)z^n$. The product of $\sum_{n\geq 0} a_n z^n$ and $\sum_{n\geq 0} b_n z^n$ is given by the so-called **convolution** of the the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$:

$$\left(\sum_{n\geq 0} a_n z^n\right)\left(\sum_{n\geq 0} b_n z^n\right) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n$$

Obviously, $\sum_{n\geq 0} a_n z^n = 0$ is the additive identity and $\sum_{n\geq 0} a_n z^n = 1$ is the multiplicative identity. The (multiplicative) **inverse** of a generating function $\sum_{n\geq 0} a_n z^n$ is a generating function $\sum_{n\geq 0} b_n z^n$ such that $\left(\sum_{n\geq 0} a_n z^n\right)\left(\sum_{n\geq 0} b_n z^n\right) = 1$.

Proposition 31 $\sum_{n\geq 0} a_n z^n$ *has an inverse if and only if* $a_0 \neq 0$.

Proof: "$\Rightarrow$": Obvious, because if $\sum_{n\geq 0} b_n z^n$ is an inverse of $\sum_{n\geq 0} a_n z^n$, then $b_0 = \frac{1}{a_0}$.
"$\Leftarrow$": Assume that $a_0 \neq 0$. By setting $b_0 = \frac{1}{a_0}$ and $b_n = -\frac{1}{a_0}\sum_{k=1}^{n} a_k b_{n-k}$ for $n \in \mathbb{N}\setminus\{0\}$ we get that $\sum_{k=0}^{n} a_k b_{n-k} = 1$ if $n = 0$ and $\sum_{k=0}^{n} a_k b_{n-k} = 0$ if $n \in \mathbb{N}\setminus\{0\}$. Thus $\left(\sum_{n\geq 0} a_n z^n\right)\left(\sum_{n\geq 0} b_n z^n\right) = 1$. $\square$

When considering generating function, we do not care of the radius of the ball where the power sum converges. However, in all our applications, the generating functions $\sum_{n\geq 0} a_n z^n$ have the property, that there is a constant $M > 0$ such that $|a_n| \leq M^n$, so at least for values $z \in \mathbb{C}$ with $z < \frac{1}{M}$, the series will converge. We always assume that z is small enough such that the series converges.

Examples: The following ways to compute generating functions are simply a consequence of the standard formula for the geometric sum:

- $\sum\limits_{n\geq 0} z^n = \frac{1}{1-z}$.
- $\sum\limits_{n\geq 0} a^n z^n = \frac{1}{1-az}$ for a constant a.
- $\sum\limits_{n\geq 0} z^{2n} = \frac{1}{1-z^2}$.

The binomial theorem implies:

- $\sum\limits_{n\geq 0} \binom{m}{n} z^n = (1+z)^m$.

By computing products of generating functions, we can get more closed formulas for generating functions. For example:

$$\frac{1}{(1-z)^2} = \left(\sum_{n\geq 0} z^n\right)\left(\sum_{n\geq 0} z^n\right) = \sum_{n\geq 0}(n+1)z^n = \sum_{n\geq 1} nz^{n-1}.$$

This leads to:

$$\sum_{n\geq 0} nz^n = \frac{z}{(1-z)^2}$$

and

$$\sum_{n\geq 0}(n+1)c^n z^n = \frac{1}{(1-cz)^2}.$$

More generally, we get:

$$\left(\frac{1}{1-z}\right)^m = \left(\sum_{n\geq 0} z^n\right)^m = \sum_{n\geq 0}\binom{m+n-1}{n} z^n$$

because $\binom{m+n-1}{n}$ is the number of ways to choose n non-distinguishable objects from m distinguishable bins.

4.1 Using Generating Functions to Solve Recursions

As an example, we consider the **Fibonacci numbers** $(F_n)_{n\in\mathbb{N}}$ which can be defined recursively by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ (for $n \in \mathbb{N} \setminus \{0,1\}$). We will show how such a homogeneous linear recursion can be solved (where "homogeneous" means that F_n is just a weighted sum of the previous numbers of the sequence without any additive term).

There are six steps to solve such a recursion:

1. State the generating function:

$$F(z) = \sum_{n\geq 0} F_n z^n$$

2. Write the generating function using the recursion:

$$F(z) = F_0 + F_n z + \sum_{n\geq 2} F_n z^n = z + \sum_{n\geq 0} F_{n+2} z^{n+2} = z + \sum_{n\geq 0}(F_{n+1} + F_n) z^{n+2}$$

3. Replace all infinite sums on the right-hand side by $F(z)$:

$$\begin{aligned}
F(z) &= z + \sum_{n\geq 0} F_{n+1} z^{n+2} + \sum_{n\geq 0} F_n z^{n+2} \quad = \quad z + z\sum_{n\geq 0} F_{n+1} z^{n+1} + z^2 \sum_{n\geq 0} F_n z^n \\
&= z + z\sum_{n\geq 1} F_n z^n + z^2 F(z) \quad = \quad z + z(F(z) - F_0 z^0) + z^2 F(z) \\
&= z + zF(z) + z^2 F(z).
\end{aligned}$$

4. Dissolve the equation for $F(z)$:

$$F(z) = \frac{z}{1 - z - z^2}.$$

5. Write the right-hand side as a formal power series.
The approach is the **partial fraction decomposition**. We search for numbers A,B,α, and β such that

$$\frac{z}{1-z-z^2} = \frac{A}{1-\alpha z} + \frac{B}{1-\beta z}$$

We can find a solution of this equation by computing a solution of the following system of equations:

(i) $(1-\alpha z)(1-\beta z) = 1 - z - z^2$

(ii) $A(1-\beta z) + B(1-\alpha z) = z$

By equating the coefficients (i) leads to $\alpha + \beta = 1$ and $\alpha \cdot \beta = -1$. By combining these equations we get $\alpha^2 - \alpha - 1 = 0$, so $\alpha \in \{\frac{1}{2} + \frac{\sqrt{5}}{2}, \frac{1}{2} - \frac{\sqrt{5}}{2}\}$. We can choose $\alpha = \frac{1}{2} + \frac{\sqrt{5}}{2}$ which implies $\beta = \frac{1}{2} - \frac{\sqrt{5}}{2}$.

Equation (ii) gives $A = -B$ and $-A\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right) - B\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) = 1$. Therefore, we get $A = \frac{1}{\sqrt{5}}$ and $B = -\frac{1}{\sqrt{5}}$.

Since these numbers A,B,α, and β solve the equations (i) and (ii), we get (by applying the formula for the geometric sum):

$$F(z) = \frac{\frac{1}{\sqrt{5}}}{1 - \left(\frac{1+\sqrt{5}}{2}\right)z} + \frac{-\frac{1}{\sqrt{5}}}{1 - \left(\frac{1-\sqrt{5}}{2}\right)z} = \sum_{n\geq 0} \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right) z^n.$$

6. Compute the sequence by equating the coefficients:

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right).$$

Remark: The number $\Phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ is called **golden ratio**.

General approach for solving linear recursions: Consider a (homogeneous) linear recursion of length k:

$$\begin{aligned} a_n &= c_1 a_{n-1} + \cdots + c_k a_{n-k} \qquad (n \geq k) \\ a_i &= b_i \qquad (i \in \{1, \ldots, k-1\}) \end{aligned}$$

1. Generating function:
$$A(z) = \sum_{n \geq 0} a_n z^n$$

2./3. Application of the recursion:
$$\begin{aligned} A(z) &= \sum_{n=0}^{k-1} b_n z^n + \sum_{n \geq k} (c_1 a_{n-1} + \cdots + c_k a_{n-k}) z^n \\ &= \sum_{n=0}^{k-1} b_n z^n + c_1 z \left(A(z) - \sum_{i=0}^{k-2} a_i z^i \right) \\ &\quad + c_2 z^2 \left(A(z) - \sum_{i=0}^{k-3} a_i z^i \right) + \cdots + c_{k-1} z^{k-1} (A(z) - a_0) + c_k z^k A(z) \end{aligned}$$

4. Dissolve the equation for $A(z)$:
$$A(z) = \frac{d_0 + d_1 z + \ldots d_{k-1} z^{k-1}}{1 - c_1 z - c_2 z^2 - \ldots c_k z^k} \qquad \text{for appropriate } d_0, \ldots, d_{k-1}$$

5. Partial fraction decomposition:
$$A(z) = \sum_{i=1}^{r} \frac{g_i(z)}{(1 - \alpha_i z)^{m_i}}$$
where $g_i(z)$ is a polynomial of degree at most $m_i - 1$ $(i = 1, \ldots, r)$.

6. Computation of the coefficients. Let $g_i(z) = \sum_{j=0}^{m_i - 1} g_{ij} z^j$ $(i = 1, \ldots, r)$. Then:
$$A(z) = \sum_{i=1}^{r} \sum_{j=0}^{m_i - 1} g_{ij} \sum_{n \geq 0} \binom{n + m_i - 1}{n} \alpha_i^n z^{n+j}.$$

Computing the partial fraction decomposition: Let $p(z) = 1 + e_1 z + e_2 z^2 + \cdots + e_k z^k$ be a polynomial with coefficient 1 or z^0. $p^R(z) = z^k + e_1 z^{k-1} + e_2 z^{k-2} + \cdots + e_k z^0$ is called the

reflected polynomial of p. This implies $p(z) = z^k p^R(\frac{1}{z})$. Let $\alpha_1, \ldots, \alpha_k$ be the (complex) zeros of p^R, so

$$p^R(z) = (z - \alpha_1) \cdot \cdots \cdot (z - \alpha_k)$$

Thus

$$p(z) = z^k \left(\frac{1}{z} - \alpha_1\right) \cdot \cdots \cdot \left(\frac{1}{z} - \alpha_k\right) = (1 - \alpha_1 z) \cdot \cdots \cdot (1 - \alpha_k z)$$

Therefore, the zeros of the reflected polynomial gives us the denominators of the partial fraction decomposition.

The numerators can be computed by comparing coefficients of the polynomials. This leads to an equation system with k variables and k equations.

Simultaneous Recursions

We can also use generating functions to solve **simultaneous recursions** of two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ where a_n may depend on $b_1, \ldots, b_{n-1}$ and b_n on $a_1, \ldots, a_{n-1}$. We consider an example that is motivated by the following question: What is the digit immediately to the right of the decimal point in the decimal representation of $(\sqrt{2} + \sqrt{3})^{1980}$?

The approach to solve this problem is to consider more generally the numbers $(\sqrt{2} + \sqrt{3})^{2n}$ for $n \in \mathbb{N}$. For small value of n, we get the following numbers:

$$\begin{array}{rcl}
(\sqrt{2} + \sqrt{3})^0 & = & 1 \\
(\sqrt{2} + \sqrt{3})^2 & = & 5 + 2\sqrt{6} \\
(\sqrt{2} + \sqrt{3})^4 & = & (5 + 2\sqrt{6})^2 \; = \; 49 + 20\sqrt{6}
\end{array}$$

Claim: There are sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ with $a_n, b_n \in \mathbb{N}$ for $n \in \mathbb{N}$ such that $(\sqrt{2} + \sqrt{3})^{2n} = a_n + b_n\sqrt{6}$.

Proof of the Claim: Apply induction: The case $n = 0$ is trivial (set $a_0 = 1$ and $b_0 = 0$). For $n \in \mathbb{N} \setminus \{0\}$, we get:

$$\begin{array}{rcl}
(\sqrt{2} + \sqrt{3})^{2n} & = & (\sqrt{2} + \sqrt{3})^{2n-2}(\sqrt{2} + \sqrt{3})^2 \\
& = & (a_{n-1} + b_{n-1}\sqrt{6})(5 + 2\sqrt{6})^2 \\
& = & (5a_{n-1} + 12b_{n-1}) + (2a_{n-1} + 5b_{n-1})\sqrt{6}
\end{array}$$

This proves the claim.

The proof also yields recursion formulas for a_n and b_n for $n \geq 1$:

$$\begin{array}{rcl}
a_n & = & 5a_{n-1} + 12b_{n-1} \\
b_n & = & 2a_{n-1} + 5b_{n-1}
\end{array}$$

Moreover, we have $a_0 = 1$ and $b_0 = 0$

We can solve this recursion by using the generating function $A(z) = \sum_{n\geq 0} a_n z^n$ and $B(z) = \sum_{n\geq 0} b_n z^n$. This gives us

$$A(z) = a_0 z^0 + \sum_{n\geq 1} a_n z^n = a_0 + \sum_{n\geq 1} (5a_{n-1} + 12b_{n-1}) z^n = 1 + 5zA(z) + 12zB(z)$$

and

$$B(z) = b_0 z^0 + \sum_{n\geq 1} b_n z^n = \sum_{n\geq 1}(2a_{n-1} + 5b_{n-1})z^n = 2zA(z) + 5zB(z).$$

The latter equation implies $B(z) = \frac{2zA(z)}{1-5z}$, and together with the previous equation, we get $A(z) = 5zA(z) + \frac{12zA(z)}{1-5z}$, so

$$A(z) = \frac{1-5z}{1-10z+z^2}$$

We use the equation

$$1 - 10z + z^2 = \left(1 - (5+2\sqrt{6})z\right)\left(1 - (5-2\sqrt{6})z\right)$$

to get a partial fraction decomposition

$$A(z) = \frac{1-5z}{1-10z+z^2} = \frac{\frac{1}{2}}{1-(5+2\sqrt{6})z} + \frac{\frac{1}{2}}{1-(5-2\sqrt{6})z}.$$

Thus

$$a_n = \frac{1}{2}\left(\left(5+2\sqrt{6}\right)^n + \left(5-2\sqrt{6}\right)^n\right). \tag{3}$$

Now, we can use this result to answer the initialquestion.

We have $(5+2\sqrt{6})^n = (\sqrt{2}+\sqrt{3})^{2n} = a_n + b_n\sqrt{6}$. Therefore, (3) implies:

$$a_n = \frac{1}{2}\left(a_n + b_n\sqrt{6} + \left(5-2\sqrt{6}\right)^n\right),$$

which leads to

$$a_n = b_n\sqrt{6} + \left(5-2\sqrt{6}\right)^n.$$

Since a_n is integral, this yields $\{b_n\sqrt{6}\} + \{(5-2\sqrt{6})^n\} = 1$ (where $\{x\} := x - \lfloor x \rfloor$ for $x \in \mathbb{R}$). However, $5 - 2\sqrt{6} < 0.11$, so for $n = 990$, the first digits after the decimal point in the decimal representation of $\left(5-2\sqrt{6}\right)^n$ are 0. Therefore, the first digits after the decimal point in the decimal representation of $b_{990}\sqrt{6}$ must be 9. Since $a_{990} \in \mathbb{N}$, the same is true for $(\sqrt{2}+\sqrt{3})^{1980} = a_{990} + b_{990}\sqrt{6}$, so the answer is "9".

4.2 Exponential Generating Functions

Definition 10 *For a sequence* $(a_n)_{n\in\mathbb{N}}$ *we call* $\hat{A}(z) = \sum_{n\geq 0} \frac{a_n}{n!} z^n$ *the* **exponential generating function** *of* $(a_n)_{n\in\mathbb{N}}$.

Thus the exponential generating function of $(a_n)_{n\in\mathbb{N}}$ is simply the generating function of $(\frac{a_n}{n!})_{n\in\mathbb{N}}$, so we can make use of all results for the generating functions. In particular, for the **product of**

the exponential generating functions $\hat{A}(z) = \sum_{n\geq 0} \frac{a_n}{n!} z^n$ and $\hat{B}(z) = \sum_{n\geq 0} \frac{b_n}{n!} z^n$ we get the exponential generating function $\hat{C}(z) = \sum_{n\geq 0} \frac{c_n}{n!} z^n$ of the sequence $(c_n)_{n\in\mathbb{N}}$ with

$$\frac{c_n}{n!} = \sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!}$$

because $(\frac{c_n}{n!})_{n\in\mathbb{N}}$ must be the convolution of $(\frac{a_n}{n!})_{n\in\mathbb{N}}$ and $(\frac{b_n}{n!})_{n\in\mathbb{N}}$.

Therefore, $\hat{C}(z) = \hat{A}(z)\hat{B}(z)$ holds if and only if for all $n \in \mathbb{N}$:

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}. \tag{4}$$

This equivalence is called **binomial convolution**.

Examples:

- By writing the Taylor series for the exponential function we get $e^{az} = \sum_{n\geq 0} \frac{a^n}{n!} z^n$. Moreover, $e^{az} \cdot e^{bz} = e^{(a+b)}z$. Hence, by using (4) and comparing the coefficients of z^n in $e^{az} \cdot e^{bz}$ and $e^{(a+b)}z$ we get again the binomial theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

- We have $(1+z)^a = \sum_{n\geq 0} \binom{a}{n} z^n = \sum_{n\geq 0} \frac{a^{\underline{n}}}{n!} z^n$, so $(1+z)^a$ is the exponential generating function of $(a^{\underline{n}})_{n\in\mathbb{N}}$. Since we have $(1+z)^a(1+z)^b = (1+z)^{a+b}$, we get by (4) and equating the coefficients

$$(a+b)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} a^{\underline{k}} b^{\underline{n-k}}$$

  By dividing this equation by $n!$ we get the Vandermonde identity:

$$\binom{a+b}{n} = \sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k}.$$

- For the derangement number D_n, we have already proved the formula $n! = \sum_{k=0}^{n} \binom{n}{k} D_k$. This means that $(n!)_{n\in\mathbb{N}}$ is the convolution of $(D_n)_{n\in\mathbb{N}}$ and the sequence $1, 1, 1, \ldots$ whose exponential generating function is $\sum_{n\geq 0} \frac{1}{n!} z^n = e^z$. Thus, with $\hat{D}(z) = \sum_{n\geq 0} \frac{D_n}{n!} z^n$ the function $\hat{D}(z) \cdot e^z$ is the exponential generating function of $(n!)_{n\in\mathbb{N}}$, so

$$\hat{D}(z) \cdot e^z = \frac{n!}{n!} z^n = \sum_{n\geq 0} z^n = \frac{1}{1-z}.$$

  This implies

$$\hat{D}(z) = \frac{e^{-z}}{1-z}. \tag{5}$$

We can consider $e^{-z} = \sum_{n\geq 0} \frac{1}{n!}(-1)^n z^n$ and $\frac{1}{1-z} = \sum_{n\geq 0} z^n$ as (standard) generating functions. By comparing coefficients in (5), this gives us once again the equation

$$\frac{D_n}{n!} = \sum_{k=0}^{n} \frac{1}{k!}(-1)^k.$$

II Graphs

5 Planar Graphs

For the lectures about planarity of graphs we refer to the textbook by Korte and Vygen [2012].

6 Colourings of Graphs

6.1 Vertex-Colourings

In this section, all graphs will be undirected.

Definition 11 *Let G be an undirected graph. A* **vertex-colouring** *of G is a mapping $c : V(G) \to \mathbb{N} \setminus \{0\}$ such that $c(v) \neq c(w)$ for every $\{v, w\} \in E(G)$. A vertex-colouring c is called* **k-vertex-colouring** *if $c(v) \leq k$ for all $v \in V(G)$. If there is a k-vertex-colouring of G, then we call G* **k-(vertex-)colourable**. *If c is a k-vertex-colouring of G, then the sets $\{v \in V(G) \mid c(v) = i\}$ are called* **colour classes** *of c ($i = 1, \ldots, k$). The* **chromatic number** *$\chi(G)$ of G is the smallest number k such that G is k-colourable. If $k = \chi(G)$, the graph G is called* **k-chromatic**.

Proposition 32 *Let G be a graph with m edges. Then*

$$\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}.$$

Proof: In a colouring with $\chi(G)$ colours there must be an edge between each pair of colour classes (otherwise we could use the same colour for both classes). Thus $m \geq \binom{\chi(G)}{2} = \frac{1}{2}\chi(G)(\chi(G) - 1)$, which is equivalent to the inequality of the proposition. $\square$

Definition 12 *The* **complement** $\bar{G}$ *of a graph* G *is the graph that is defined by the vertex set* $V(\bar{G}) := V(G)$ *and the edge set* $E(\bar{G}) := \binom{V(\bar{G})}{2} \setminus E(G)$.

Proposition 33 *(Nordhaus and Gaddum [1956]) For every graph* G *with* $|V(G)| = n$ *we have:*

$$\begin{array}{llcll} (a) & 2\sqrt{n} & \leq & \chi(G) + \chi(\bar{G}) & \leq & n+1, \\ (b) & n & \leq & \chi(G)\chi(\bar{G}) & \leq & \left(\frac{n+1}{2}\right)^2. \end{array}$$

Proof: (i) Let $c : V(G) \to \{1, \dots, \chi(G)\}$ be a vertex-colouring of G with $\chi(G)$ colours. Let n_i be the number of vertices of G coloured with i ($i = 1, \dots, \chi(G)$). Then, $\max_{i \in \{1,\dots,\chi(G)\}} n_i \geq n/\chi(G)$. Since all vertices of a colour class of c have to have different colours in a vertex-colouring of $\bar{G}$, we get $\chi(\bar{G}) \geq \max_{i \in \{1,\dots,\chi(G)\}} n_i$. This implies $\chi(\bar{G}) \geq n/\chi(G)$ and thus $\chi(G)\chi(\bar{G}) \geq n$.

(ii) The inequality $(\chi(G) - \chi(\bar{G}))^2 \geq 0$ implies $(\chi(G) + \chi(\bar{G}))^2 \geq 4\chi(G)\chi(\bar{G})$, and hence we get $\chi(G) + \chi(\bar{G}) \geq 2(\chi(G)\chi(\bar{G}))^{\frac{1}{2}} \overset{(i)}{\geq} 2\sqrt{n}$.

(iii) We show $\chi(G) + \chi(\bar{G}) \leq n + 1$ by induction in $n = |V(G)|$.
For $n = 1$, the statement is trivial.
Let $n > 0$ and $v \in V(G)$. By induction hypothesis we have $\chi(G - v) + \chi(\bar{G} - v) \leq n$. Moreover

$$\chi(G) \leq \chi(G - v) + 1$$

and

$$\chi(\bar{G}) \leq \chi(\bar{G} - v) + 1.$$

If at least of one of the last two inequalities is a strict inequality, then the statement follows directly. Hence assume that $\chi(G) = \chi(G - v) + 1$ and $\chi(\bar{G}) = \chi(\bar{G} - v) + 1$. This gives $|\delta_G(v)| \geq \chi(G - v)$ and $|\delta_{\bar{G}}(v)| (= n - 1 - |\delta_G(v)|) \geq \chi(\bar{G} - v)$. Since $|\delta_G(v)| + |\delta_{\bar{G}}(v)| = n - 1$, we get $\chi(G - v) + \chi(\bar{G} - v) \leq n - 1$ and finally $\chi(G) + \chi(\bar{G}) \leq n + 1$.

(iv) The inequality $\chi(G)\chi(\bar{G}) \leq \left(\frac{n+1}{2}\right)^2$ follows form the inequality shown in step (iii) and the inequality $(\chi(G) + \chi(\bar{G}))^2 \geq 4\chi(G)\chi(\bar{G})$. □

Proposition 34 *For every graph* G*, we have:*

$$\chi(G) \leq \Delta(G) + 1.$$

Proof: The following greedy-algorithm computes a colouring with $\Delta(G) + 1$ colour: Traverse the vertices of G in an arbitrary ordering and colour each vertex v with the first colour that has not yet been used at a neighbour of v. □

Of course, the proof of the previous proposition yields an algorithm to compute a vertex-colouring with $\Delta(G)+1$ colours. However the chromatic number of a graph can be much smaller than $\Delta(G)$, see for example the graph $K_{1,n-1}$ where $\chi(K_{1,n-1}) = 2$ but $\Delta(K_{1,n-1}) = n-1$.

Examples for graphs G with $\chi(G) = \Delta(G) + 1$ are complete graphs and odd cycles. For all other connected graphs we get a better bound on $\chi(G)$:

Theorem 35 *(Brooks' Theorem, Brooks [1941]) Let G be a connected graph that is neither a complete graph nor a cycle of odd length. Then:*

$$\chi(G) \leq \Delta(G).$$

Proof: Assume that the statement is false. Let G be a smallest (with respect to the number of vertices) counterexample, so in particular G is connect but neither a complete graph nor a cyle of odd length, and we have $\chi(G) = \Delta(G) + 1$. This implies $\Delta(G) > 2$ because G cannot be a cycle of even length or a path (in that case, we would have $\chi(G) = \Delta(G) = 2$).

Choose a vertex v with $|\delta_G(v)| = \Delta(G)$. Since G is not a complete graph, there must be two neighbours u and w of v that are not connected by an edge in G. We distinguish two cases:

Case 1: $G[V(G) \setminus \{u, w\}]$ is not connected.
Let A_1 be a connected component of $G[V(G) \setminus \{u, w\}]$. Set $V_1 := V(A_1) \cup \{u, w\}$ and $V_2 := V(G) \setminus V(A_1)$.
Then $\chi(G[V_i]) \leq \Delta(G)$ because $G[V_i]$ is not complete (there is no edge between u and w) and $\Delta(G) \geq 3$ ($i \in \{1, 2\}$). If both for $G[V_1]$ and $G[V_2]$ there a vertex-colouring with $\Delta(G)$ colours such that u and w get different colours then we can choose these two colourings in such a way that u gets in both of them colour 1 and w gets in both of them colour 2. This way, we receive a vertex-colouring of G with $\Delta(G)$ colours. Thus, we can assume that there is a $j \in \{1, 2\}$ such that every $\Delta(G)$-vertex-colouring of $G[V_j]$ colours u and w with the same colour. Then both u and w have degree at least $\Delta(G) - 1$ in $G[V_j]$. Hence in $G[V_{3-j}]$ they have degree at most 1. Thus, as $\Delta(G) \geq 3$, also $G[V_{3-j}]$ has a vertex-colouring with $\Delta(G)$ colours where u and v get the same colour. Therefore G is $\Delta(G)$-colourable.

Case 2: $G[V(G) \setminus \{u, w\}]$ is connected.
Then $G[V(G) \setminus \{u, w\}]$ contains a spanning tree T. Colour u and w with colour 1. Afterwards perform $(n-3)$-times the followings steps:

(1) Choose a leaf x of T with $v \neq x$.

(2) Colour x with the smallest colour not used at the neighbours of x in G.

(3) Remove x from T.

After these steps all vertices except v have been coloured. For this colouring we need at most $\Delta(G)$ colours since by the choice of x step (1) the vertex x always has an un-coloured neighbour (namly its neighbour in T). Thus, when x is coloured at most $\Delta(G) - 1$ of its neightbors have already been coloured.

Finally, we have to assign a colour to v. All of its $\Delta(G)$ neighbours have already been coloured but at least two of them (u and w) got the same colour, so also for v we can choose one of the colours in $\{1, \ldots, \Delta(G)\}$. $\square$

Notation: Let G be a graph. We denote by $\alpha(G)$ the size of a maximum stable set in G, i.e. the size of a largest set X of vertices of G such that $G[X]$ does not contain any edges. By $\omega(G)$, we denote the size of a largest clique in G, i.e. the size of a largest set X of vertices such that $G[X]$ is a complete graph.

Satz 36 *For every $k \in \mathbb{N} \setminus \{0\}$ there is a graph G_k with $\chi(G_k) = k$ and $\omega(G_k) \leq 2$.*

Proof: The graphs G_k can be built recursively. For $k = 1$, this is trivial.

Hence, let $k > 1$. We assume that the graphs $G_1, \ldots, G_{k-1}$ have already been built. G_k contains a copy of each of the graphs $G_1, \ldots, G_{k-1}$ as a subgraph. In addition G_k contains a vertex set A_k consisting of $|V(G_1)| \cdot |V(G_2)| \cdot \cdots \cdot |V(G_{k-1})|$ vertices. Choose a bijection $\tau_k : A_k \rightarrow \{(v_1, \ldots, v_{k-1}) \mid v_1 \in V(G_1), \ldots, v_{k-1} \in V(G_{k-1})\}$. Then G_k contains (apart from the edges in the subgraphs $G_1, \ldots, G_{k-1}$) for each vertex $v \in A_k$ an edge from v to the elements of the $(k-1)$-tupel $\tau_k(v)$. By construction, the graph G_k does not contain any cycles of length three (provided that the graphs $G_1, \ldots, G_{k-1}$ do not contain any 3-cycle).

Under the assumption that each G_i with $i \in \{1, \ldots, k-1\}$ can be coloured with i colours, the graph G_k can be coloured with k-colours: for the colouring of all subgraphs $G_1, \ldots, G_{k-1}$ we need in total $k-1$ colours and the vertices in A_k (which is a stable set) can get the same colour.

On the other hand, G_k is not $(k-1)$-colourable if none of the G_i $(i = 1, \ldots, k-1)$ is $(i-1)$-colourable. To prove this, assume that there was a $k-1$-colouring of G_k. Then choose a vertex v_1 in G_1 with colour c_1. There must be a vertex v_2 in G_2 with colour $c_2 \neq c_1$ because G_2 cannot be coloured with just one colour. Since G_3 is not 2-colourable there must be a vertex v_3 ind G_3 with colour $c_3 \notin \{c_1, c_2\}$. We can continue this and get for each $i \in \{1, \ldots, k-1\}$ a vertex v_i in G_i whose colour c_i is not contained in $\{c_1, \ldots, c_{i-1}\}$. But there is a vertex $v \in A_k$ with $\tau_k(v) = (v_1, \ldots, v_{k-1})$. Thus, v is in G_k connected by an edge to all vertices in $\{v_1, \ldots, v_{k-1}\}$, so it cannot be coloured with any colour from $c_1, \ldots, c_{k-1}$. Therefore, we need k colours for a vertex-colouring of G_k. $\square$

Remark: For any $k \in \mathbb{N}$ there are graphs G with $\chi(G) \geq k$ that do not contain any cycle of length less than k (see Diestel [2005] for proof of this statement).

Definition 13 *An undirected graph G is called* **perfect** *if $\chi(H) = \omega(H)$ hold for every induced subgraph H of G.*

There are several NP-hard problems that can be solved in polynomial-time if we restrict the instances to perfect graphs. For exampe, one can compute maximum stable sets, maximum cliques and optimum vertex-coloring in perfect graphs in polynomial time (see Grötschel, Lovász und Schrijver [1984]).

Satz 37 *A graph is perfect if and only* $\alpha(H)\omega(H) \geq |V(H)|$ *holds for every induced subgraph* H*.*

Proof: The proof is taken from Schrijver [2003].

„$\Rightarrow$“ Let G be perfect and H an induced subgraph of G. Then $\chi(H) = \omega(H)$, and since $\alpha(H)\chi(H) \geq |V(H)|$, this implies $\alpha(H)\omega(H) \geq |V(H)|$.

„$\Leftarrow$“ Assume that there is a non-perfect graph G such that $\alpha(H)\omega(H) \geq |V(H)|$ for each induced subgraph H of G. We can assume that G is a smallest graph with this property, so in particular any induced subgraph of G is perfect.

Let $V(G) = \{1, \dots, n\}$, $\alpha = \alpha(G)$ and $\omega = \omega(G)$.

We first show that there are stable sets $S_0, S_1, \dots, S_{\alpha\omega}$ in G such that every vertex is contained in exactly one of these α sets.

Let S_0 be a stable set of size α. For every $v \in S_0$ the graph $G - v$ is perfect, so $\chi(G - v) = \omega(G - v) \leq \omega(G)$. Thus $V(G) \setminus \{v\}$ can be decomposed in ω stable sets. By doing this for every $v \in S_0$, we get the sets $S_0, S_1, \dots, S_{\alpha\omega}$.

For each set S_i ($i = 0, \dots, \alpha\omega$) there is a clique C_i of size ω with $C_i \cap S_i = \emptyset$, because otherwise we had $\omega(G) \geq \omega(G - S_i) + 1 = \chi(G - S_i) + 1 \geq \chi(G)$, which means that G is perfect.

Every clique C_i and every stable set S_j have at most one vertex in common, but in total every C_i intersects $\alpha\omega$ of the sets S_j, because each of the ω elements of C_i is contained in α of fthe stable sets. Therefore, $|C_i \cap S_j| = 1$ for $i \neq j$.

Consider two $(\alpha\omega + 1) \times n$-inzidence matrices M uad N with entries 0 and 1. Let $M = (M_{ij})_{(i,j)\in\{0,\dots,\alpha\omega+1\}\times\{1,\dots,n\}}$ with $M_{ij} = 1 \Leftrightarrow j \in S_i$ and $N = (N_{ij})_{(i,j)\in\{1,\dots,\alpha\omega+1\}\times\{1,\dots,n\}}$ with $N_{ij} = 1 \Leftrightarrow j \in C_i$.

Then, $MN^t = J - I$ where J is an $(\alpha\omega + 1) \times (\alpha\omega + 1)$-matrix consisting of ones only, and I is the $(\alpha\omega + 1) \times (\alpha\omega + 1)$-identity matrix. The matrix $J - I$ has rank $\alpha\omega + 1$, which implies $n \geq \alpha\omega + 1$. This is a contradiction to our assumption that $\alpha(H)\omega(H) \geq |V(H)|$ for each induced subgraph H of G. $\square$

Corollary 38 *(Weak perfect graph theorem) A graph* G *is perfect if and only if* $\bar{G}$ *is perfect.*

Proof: Follows directly from the previous theorem. □

Theorem 39 *(Strong perfect graph theorem) (Chudnovsky et al. [2006]) A graph G is perfect if and only if it does not contain an odd cycle with length at least 5 nor the complement of an odd cycle with length at least 5 as an induced subgraph.*

For a proof we refer to Chudnovsky et al. [2006].

Theorem 40 *It can be checked in time $O(|V(G)|^9)$ if a given graph G is perfect.*

For a proof see Chudnovsky et al. [2005].

Theorem 41 *For every planar graph G, we have $\chi(G) \leq 5$.*

For a proof, we refer to Korte and Vygen [2012].

Remark: By the famous (**Four Colour Theorem**) even for colour suffice to colour planar graphs, so $\chi(G) \leq 4$ for any planar graph G. The first proof of this theorem was given in 1977 by Appel and Haken (Appel and Haken [1977a], Appel and Haken [1977b]). The proof is quite involved and cannot be shown here. The proof is based on the analysis of so-called *configurations*. A configuration is a connected subgraph of a graph G where we are given in addition degrees of the vertices of the subgraph in G. A set M of configurations is called *unavoidable* is every planar graph contains a configuration in M. We know for example that the set M^* consisting of 6 copies of K_1, where we set the degree of the vertex in the i-th copy to i ($i = 0, \ldots, 5$), is unavoidable because every planar graph has a vertex of degree at most 5.

A configuration is called *reducible* if no smallest counter example to the Four Colour Theorem contains it. For example the configuration which consists of the graph K_1 where we demand a vertex degree of 3 is obviously reducible. Now the goal is to find an unavoidable set of reducible configurations. For the Five Colour Theorem the elements of M^* are reducible. For the Four Colour Theorem, Appel and Haken could find an unavoidable set of reducible configurations that consists of 1936 elements To check this set of configurations they had to use a computer. Later on, they could reduce the number of configurations to 1476. A somewhat shorter proof that nedded only 633 configuration (but nevertheless needed the help of a computer) was given by Robertson et al. [1997]. The ideas of the proof of the Four Colour Theorem are summarized by Woodall und Wilson [1978] (see also Bollobás [1979]).

6.2 List-Colourings

Definition 14 *Let G be a graph. Assume that for every vertex $v \in V(G)$ we are given a set C_v (the* **colour-list** *of v). A (feasible)* **vertex-list-colouring** *is a mapping $c : V(G) \to \cup_{v \in V(G)} C_v$ such that $c(v) \in C_v$ for every vertex $v \in V(G)$ and $c(v) \neq c(w)$ for every edge $\{v, w\} \in E(G)$ gilt. The* **list-chromatic number** *$\chi_l(G)$ of G is the smallest number such that for any choice of colour-lists of length at least $\chi_l(G)$ a vertex-list-colouring exists.*

Observation: For every graph G we have $\chi(G) \leq \chi_l(G)$.

Proposition 42 *For $k \in \mathbb{N}$ we have $\chi_l(K_{k,k^k}) > k$.*

Proof: Let $V(K_{k,k^k}) = A_k \dot{\cup} B_k$ with $|A_k| = k$, $|B_k| = k^k$ and $E(K_{k,k^k}) = \{\{a, b\} \mid a \in A_k, b \in B_k\}$. We choose all colours for the lists C_v (für $v \in V(G)$) from a set $\{1, \dots, k^2\}$. Let $A_k = \{a_1, \dots, a_k\}$. Set $C_{a_i} = \{(i-1)k + 1, (i-1)k + 2, \dots, (i-1)k + k\}$ (for $i \in \{1, \dots, k\}$). Thus, the sets C_{a_i} are pairwise disjoint. Chose a bijection

$$\phi : \{1, \dots, k^k\} \to \Big\{X \subseteq \{1, \dots, k^2\} \mid |X| = k, |X \cap C(a_i)| = 1 \text{ for all } i \in \{1, \dots, k\}\Big\}.$$

With $B_k = \{b_1, \dots, b_{k^k}\}$ we set $C_{b_j} = \phi(j)$ $(j \in \{1, \dots, k^k\})$. If we colour each element $a_i \in A_k$ with a colour $c_i \in C_{a_i}$ $(i \in \{1, \dots, k\})$, then for b_j with $\phi(j) = \{c_1, \dots, c_k\}$ there is no colour left. Hence there is no vertex-list-colouring of K_{k,k^k} for these lists and therefore $\chi_l(K_{k,k^k}) > k$. $\square$

Theorem 43 *For every planar graph G we have $\chi_l(G) \leq 5$.*

Proof: We can assume that G is connected and that there is a planar embedding of G such that all boundaries of regions are cycles and that for all region with the possible exception of the unbounded region these cycles have length 3. (swe call such graphs *nearly triangulated*). If these conditions aren't met we can add edges until they are met (and adding edges can only increase the list chromatic number).

We show the theorem by proving the following statement by induction in the number of vertices:

Let G be a nearly triangulated planar graph with fixed planar embedding. Let B be the cycle on the boundary of the unbounded region. We are given colour lsist C_v ($v \in V(G)$) wit hthe following properties:

- *There are two vertices x and y that are neighbours on B and two different colour α and β such that $C_x = \{\alpha\}$ and $C_y = \{\beta\}$.*

- *For all other vertices v on B we have $|C_v| \geq 3$.*
- *For the vertices v not contained in B we have $|C_v| \geq 5$.*

Then, there is a list colouring of G for these colour lists.

We apply induction in the number of vertices. First assume $|V(G)| = 3$. Then the statement is valid because apart from x and y there is only one vertex v in G, and for this vertex we have $|C_v| \geq 3$, so the colour list C_v of V contains a colour that is different from α and β.

Fof $|V(G)| > 3$ we distinduish two cases:

Case 1: G contains an edge that is a chord of B, i.e. an edge connecting two vertices u and v on B such that u are v are not neighboured on B.

Then, B contains two different u-v-paths B_1 und B_2. For $i \in \{1, 2\}$ let G_i be the subgraph G that is bounded by the embedding of B_i and $\{u, v\}$. Thus, G_1 and G_2 have exactly u und v as common vertices. W.l.o.g. we can assume that B_1 contains the vertices x and y. Now we apply the induction hypothesis to G_1 and get a listen colouring of G_1. Then, in particular u and v have been coloured. G_2 (with the colour list of u and v reduced to one element each) fulfills all conditions of the statement and we have $|V(G_2)| \leq |V(G)|$. Hence we can also apply the imduction hypothesis to G_2 and extend the list colouring of G_1 to a list colouring of G.

Case 2: G does not contain a chord of B.

Let v be the neighbour of x on B that is different from y. Let w be the neighbour of v on B that is different from x. Thus B contains the edges $\{w, v\}$, $\{v, x\}$, and $\{x, y\}$ ($w = y$ is possible). Let X be the set of neighbours of v. Since G is nearly triangulated, the graph $G' := G - v$ is nearly triangulated, too. As $|C_v| \geq 3$, the set C_v contains at least colours γ and δ that are different from α. Now remove for each vertex $z \in X \setminus \{w, x\}$ the colours γ and δ from the colour list C_z. Since all elements of $X \setminus \{w, x\}$ are on the outer boundary of G' but not on the outer boundary of G (otherwise G would contain a chord of B), G' meets all conditions of the statement that we want to prove. We can apply the induction hypothesis to G' an get a list colouring of G' where of all vertices in X at most w is coloured with one of the colours γ or δ. Thus we can extend this solouring to a list colouring of G by assigning to v one of the colours γ and δ that has not been used for w. $\square$

Remark: This bound on the list chromatic number of G is best possible since there are planar graphs G with $\chi_l(G) = 5$ (see the exercises).

6.3 Edge-Colourings

In this section, we assume that all graphs are simple.

Definition 15 *Let G be an undirected graph. A (feasible) k-*edge-colouring *of G is a mapping $c : E(G) \to \{1, \ldots, k\}$ such that $c(e_1) \neq c(e_2)$ for every two edges $e_1, e_2 \in E(G)$ with $|e_1 \cap e_2| = 1$. If there is a feasible k-edge-colouring then G is called k-*edge-colourable*. The sets $\{e \in E(G) \mid c(e) = i\}$ are called* colour classes *of the edge-colouring c ($i = 1, \ldots, k$). The* chromatic index *$\chi'(G)$ of G is the smallest number k such that G is k-edge-colourable.*

Theorem 44 *(Vizing's Theorem) For every undirected graph G we have $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

For a proof we refer to the textbook by Korte and Vygen [2012].

Theorem 45 *(König's Theorem) For every bipartite graph G we have $\chi'(G) = \Delta(G)$.*

Proof: We prove the statement by induction in $m = |E(G)|$. For $m = 0$, the statement is trivial.

Thus, let $m > 0$ and let $e = \{v, w\} \in E(G)$ be an edge. By Induction hypothesis, $G - e$ has an edge-colouring $c : E(G) \setminus \{e\} \to \{1, \ldots, \Delta(G)\}$. In $G - e$ the vertices v and w have at most $\Delta(G) - 1$ neighbours, so there are numbers $n(v), n(w) \in \{1, \ldots, \Delta(G)\}$ such that $n(v) \neq c(e')$ for all edges $e' \in (\delta_G(v) \setminus \{e\})$ and $n(w) \neq c(e')$ for all edges $e' \in (\delta_G(w) \setminus \{e\})$. If $n(v) = n(w)$, we can colours $\{v, w\}$ with the colour $n(v)$ and are done. Hence assume $n(v) \neq n(w)$. Consider the subgraph $H = (V(G), \{e' \in E(G) \mid c(e') \in \{n(v), n(w)\}\})$. All vertices in H have degree at most 2, and v has degree at most 1. Consider a longest path P in H starting in v. The edges of the path are alternately coloured with $n(w)$ and $n(v)$. The path P cannot end in w enden, because in that case its last edge was an $n(v)$-edge and P together with the edge $\{v, w\}$ would be a cycle of odd length (in contradiction to the assumption that G is bipartite). Hence, we can swap the colours $n(v)$ and $n(w)$ on P and colour the edge $\{v, w\}$ with the colour $n(w)$. □

Notation: Let G be a graph. For $k \in \mathbb{N}$, we call G **k-regular** if all vertices in g are of degree k. We call an edge $e \in E(G)$ a **bridge** if $G - e$ contains more connected components than G.

Theorem 46 *Let G be a 3-regular planar graph without bridges. Then $\chi'(G) = 3$.*

Proof: Since $\chi'(G) \geq 3$ is trivial, we only have to show $\chi'(G) \leq 3$.

W.l.o.g. we can assume that G is connected (otherwise consider the connected components of G).

Consider a fixed planar embedding of G. The Four Colour Theorem implies that we can colour the faces of the embedding with four colours such that neighbouring faces get different colours. Since G does not contain a bridge, every edge is on the boundary of exactly two faces. For an edge e let C_e be the set of the (two) colours of the faces that are bounded by e. Then we set

$$c(e) := \begin{cases} 1 & : \quad C_e = \{1,2\} \text{ or } C_e = \{3,4\} \\ 2 & : \quad C_e = \{1,3\} \text{ or } C_e = \{2,4\} \\ 3 & : \quad C_e = \{1,4\} \text{ or } C_e = \{2,3\} \end{cases}$$

Every vertex touches exactly three faces, so by this assignment no two edges that are incident to the same vertex can get the same colour. $\square$

We applied the Four Colour Theorem to proves the previous theorem. On the other hand, the For Colur Theorem can ve proven easily by using the previous theorem. First of all observe the for proving the Four Colour Theorem it is sufficient to prove that for any planar embedding of a planar graph G the faces of the embedding can be coloured with four colours such that neighbouring facs get different colours. Moreover we can assume that G is connected, does not contain any bridges and is 3-regular. By the previous theorem such a graphs has a feasible edge colouring $c : E(G) \to \{1,2,3\}$. In order to colour the faces of a planar embedding of G with four colours, it is sufficient to assign to each face A of the embedding of G an ordered pair (α_A, β_A) with $\alpha_A, \beta_A \in \{1,2\}$ such that for each two neighbouring faces A and B we have $(\alpha_A, \beta_A) \neq (\alpha_B, \beta_B)$. The numbers α_A and β_A can be found in the following way. For $(i, j \in \{1,2,3\}$, let $H_{i,j} := (V(G), \{e \in E(G) \mid c(e) \in \{i,j\}\})$. For every two colours i and j from $\{1,2,3\}$ the graph $H_{i,j}$ is a 2-regular graph for which we are given a planra embedding by the planar embedding of G. Obviously, there is a geasible colouring $\tilde{c}_{i,j}$ of the faces of the planar embedding of $H_{i,j}$ with two colours, Any face A of he embedding of G belongs two exactly one face $A_{1,2}$ of the embedding of $H_{1,2}$ and two one face $A_{1,3}$ of the embedding of $H_{1,3}$. We set $\alpha_A := \tilde{c}_{1,2}(A_{1,2})$ and $\beta_A := \tilde{c}_{1,3}(A_{1,3})$. It is easy to check that the pairs (α_A, β_A) have the desired properties.

Remark: The equivalence of the Four Colour Theorem and Theorem 46 was known before the For Colour Theorem has been proved.

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